The minimum number of Fox colors modulo 13 is 5

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Abstract

In this article we show that if a knot diagram admits a non-trivial coloring modulo 13 then there is an equivalent diagram which can be colored with 5 colors. Leaning on known results, this implies that the minimum number of colors modulo 13 is 5.

Keywords: knots, links, colorings, minimum number of colors.

MSC 2010: 57M27

1 Introduction

The Fox colorings of a knot or link ([2], exercise 6 on page 92) are the solutions of a system of linear homogeneous equations read off a diagram of the knot or link at issue. Arcs of the diagram are envisaged as algebraic variables and at each crossing of the diagram the equation "twice the over-arc minus the under-arcs equals zero" is read. The coefficient matrix of this system of equations is called **coloring matrix** of the diagram under study. It has the following feature. Along each row there are exactly one 2, two -1's and the rest of the entries are 0's. It follows that the determinant of the coloring matrix is 0. Furthermore, upon performance of a Reidemeister move on the diagram, the coloring matrix corresponding to the new diagram relates to the coloring matrix corresponding to the former diagram by elementary transformations on matrices. Thus the equivalence class of the coloring matrix for any diagram of the knot under study is an invariant. Let us choose for representative of this equivalence class the Smith Normal Form (SNF) of the coloring matrix. Since the determinant of these matrices is 0, then one of the entries of the diagonal of the SNF is 0 and this corresponds to the monochromatic solutions i.e., the solutions obtained by assigning the same color (number) to each arc of the diagram. Polichromatic solutions, also known as non-trivial solutions, are obtained if there is at least one more 0 along the diagonal of the SNF. In general, especially in the case of knots and non-split links, this involves working over the modular integers for a specific prime modulus p. If our knot or link, L, admits non-trivial colorings over a modulus p, Harary and Kauffman, [4], introduced the minimum number of colors of L mod p, notation $mincol_p(L)$ to be the minimum number of distinct colors it takes to assemble a non-trivial coloring mod p, the minimum being taken over all diagrams of the link at issue.

At this point we warn the reader that any knot or link considered in this article has non-zero determinant, the **determinant** of the knot or link being the product of the entries of the diagonal of its SNF

but the 0 referred to above. As a matter of fact, a link with zero determinant is colorable modulo any prime which makes these links quite special and deserving a separate article.

There is a number of articles on the topic of minimum number of colors [1, 3, 5, 6, 7, 8, 9, 10, 11, 12, 13]. In [13], Satoh developed a technique for finding the minimum number of colors over a fixed modulus but on an otherwise arbitrary situation. One considers a diagram equipped with a non-trivial coloring on the given modulus and using all available colors in all possible ways. The idea is then to remove one color at a time until one cannot remove any more colors. To remove one color one assumes it shows up in the diagram in all possible ways. Specifically, we assume it shows up as the color of a monochromatic crossing and devise a procedure to eliminate that color from this monochromatic crossing; we repeat the procedure for each monochromatic crossing bearing this color. These will be called the α instances. Then we assume the color at stake shows up at the over-arc of a polichromatic crossing and devise a procedure to eliminate it from the over-arc; and we repeat the procedure for all polichromatic crossings with this color on the over-arc. We call these the β instances. Finally we assume the color shows up on an under-arc connecting two crossings, and devise a procedure to eliminate it and repeat it over all such situations. Here we distinguish two cases. If the adjacent over-arcs bear distinct colors we call them γ instances; otherwise δ instances. In each of the α, β, γ, and δ instances, the procedure for eliminating the color consists of performing Reidemeister moves (accompanied by the unique rearrangement of colors that yields a coloring in the new diagram) so that the color at issue is eliminated. The transformations or sequences of Reidemeister moves that take care of α instances will be called α_i's and analogously for the other instances. If color c has been successfully eliminated in each of the α, β, γ, and δ instances, one moves on to color c' and iterates the procedure taking into consideration this time that color c is no longer there (nor the colors previously removed). Satoh applied this technique successfully in [13] to show that mod 5 four colors suffice. Then Oshiro [11] made the first impressive use of this technique by eliminating a string of 3 colors mod 7 thus showing that mod 7, 4 colors suffice. Using the same technique, Cheng et al [1] proved that at most 6 colors are needed mod 11, and Nakamura et al [10] proved further that 5 is the minimum number of colors for any knot or link admitting non-trivial 11-colorings. In the current article we apply Satoh's technique to prove the following result.

Theorem 1.1. *For any knot or link (with non-zero determinant) admitting non-trivial colorings modulo* 13*, there is a diagram of it equipped with a non-trivial coloring modulo* 13 *using* 5 *colors.*

Corollary 1.1. *If L is a knot or link in the conditions of Theorem 1.1, then*

$$mincol_{13}\, L = 5.$$

Furthermore, since there is essentially one set of 5 *colors modulo* 13 *which can color a non-trivial coloring, there is a Universal* 13*-Minimal Sufficient Set of Colors, in the sense of* [9].

Proof. Since it is known ([10, 8, 5]) that the minimum number of colors modulo 13 has to be at least 5 then the equality follows from the Theorem.

Furthermore, it is also known ([3], see also [9]) that there is only one equivalence class of colors of cardinality 5 modulo 13. Then the Theorem proves that any such coloring set of 5 colors colors any diagram which admits a minimal coloring modulo 13. ∎

We prove Theorem 1.1 as follows. The sequence of colors to be removed is $12, 11, 6, 3, 4, 8, 9, 2$. We organize the removal of these colors into three parts. In the first one, Section 2, we remove colors 12 and 11 for few transformations are needed to eliminate these colors, with very few exceptions. In the second part, Section 3, we remove colors $6, 3, 4$ and 8. Here a considerable number of transformations is needed. Finally, in the third part, Section 4, we remove colors 9 and 2. Here, due to the fact that many colors have already been removed, the parameters can take on but a few values. Thus few transformations are needed in general and the really exceptional cases require a different approach.

Remark Any equality is to be understood modulo 13, in the rest of this article.

1.1 Acknowledgements

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2 Part I: Elimination of colors 12 and 11.

In this Section we remove colors 12 and 11. For each of the instances α, β, γ, and δ there is one or two transformations that take care of most of the cases. Our strategy here is to force the unwanted situations and then prove that either they cannot occur or they are taken care of by one of the alternative transformations. Here are a couple of illustrative examples. The first one is the elimination of monochromatic crossings, with color $c = 12$. Since our working coloring is non-trivial and the knot or link has non-trivial determinant, then if there are monochromatic crossings with color $c = 12$, one of them has an adjacent crossing which looks like the left-hand side of transformations α_1 (or α_2) or α_3, where $a \neq 12$ (Figure 1). (We know $a \neq 12$ since otherwise the coloring would be trivial.) Since $2a + 1$ is our current $2a - c$ (since we assumed $c = 12$) then transformations α_1 and α_3 guarantee the elimination of this monochromatic crossing provided $2a + 1 \neq 12$ since 12 is precisely the color we are trying to remove. So forcing the unwanted situation here is to state $2a + 1 = 12$. But the only solution is $a = 12$ which as discussed above cannot occur. So we conclude that transformations α_1 and α_3 suffice for eliminating the monochromatic crossings with color 12. Now for a more elaborate example. The elimination of color $c = 12$ from an over-arc of a polichromatic crossing, assuming there are no more monochromatic crossings with color $c = 12$ left. The situation is depicted on the left-hand side of the top row of Figure 2. Thus $a \neq 12$ and if $2a + 1 \neq 12$ and $3a + 2 \neq 12$ then transformation β_1 does the job. So forcing the unwanted situations here is to state $2a + 1 = 12$ or $3a + 2 = 12$. But the solution to either of these equations is $a = 12$ which cannot occur. We then conclude that the elimination of polichromatic crossings whose over-arcs bear color 12 is realized by transformation β_1. Considering the left-hand side of Figure 2 again we could argue that we also do not have to consider values of a such that $24 - a = 12$. The fact is that our analysis of $2a + 1 = 12$ or $3a + 2 = 12$ allows us to disregard such details, in this case. Furthermore, it should be noted that in both situations we assumed (as we should) that 12 was the first color to be removed i.e., no other color had already been removed. Subsequent cases might not be so simple. We now proceed with the analysis of the other instances without this much detail.

For each of the α, β, γ and δ instances, we will treat the elimination of 12 and 11; it will be clear to the reader that this is equivalent to treating the elimination of 12 in the $\alpha, \beta, \gamma, \delta$ instances followed by treating the elimination of 11.

1. The α instance.

   In order to remove color $c = 12$ and $c = 11$ from a monochromatic crossing, consider the transformations α_1 in Figure 1.

   (a) $c = 12$: ditto.

   (b) $c = 11 = -2$

   Here the unwanted situations are $2a - 11 = 11$ or $2a - 11 = 12$. In the first case we have $a = 11$ which cannot occur. In the second case we have $2a = 23 = 10$ and so $a = 5$. In this case transformation α_2 takes care of the issue since $3c - 2a = 36 - 10 = 0$.

2. The β instance.

   Consider transformation β_1 in Figure 2.

   (a) $c = 12$: ditto.

   (b) $c = 11$.

   Then if either $2a - 11 = 11$ or $3a - 22 = 11$ then $a = 11$ which contradicts the working hypothesis. So let us look into $2a - 11 = 12$ and/or $3a - 22 = 12$. The solutions are $a = 5$ and $a = 7$. Then $2c - a = 4$ and $2c - a = 2$, respectively, and transformation β_2 in Figure 2 takes care of these issues.

3. The γ instance.

   Consider transformation γ_1 in Figure 3.

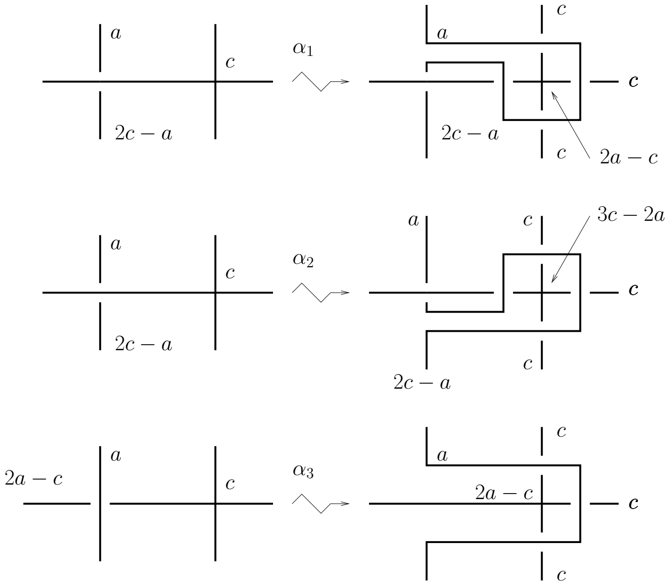

Figure 1: Transformation α_1, α_2, and α_3. The c $(2a-c$ or $3c-2a)$ close to the crossing means this crossing is monochromatic with the indicated color. The details of these crossings are not specified for they do not matter for the argument. The following remark is in oreder for α_3. Either $2a-c$ has already been removed or is currently being removed and the left hand-side of α_3 will never materialize in our diagram. Or $2a-c$ is still available and α_3 trivially eliminates the monochromatic crossing colored with color c.

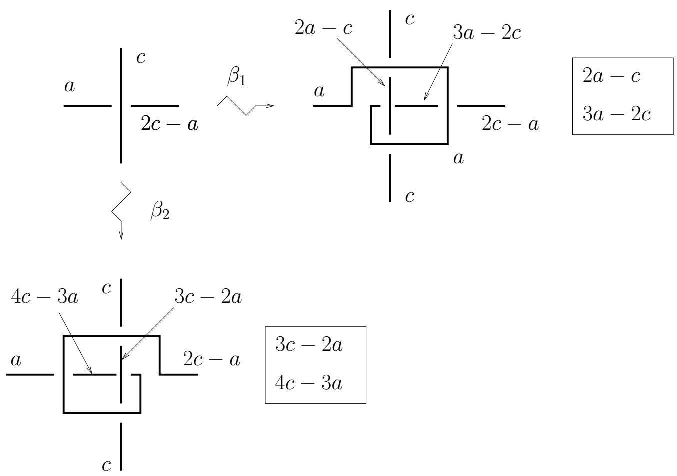

Figure 2: Transformations β_1 and β_2.

(a) $c=12$

Then $2a-2b+c=2a-2b-1$ and if $2a-2b-1=-1$ then $2(a-b)=0$ and so $a=b$ which is contrary to the assumptions.

If $2a-b=-1$ then $b=2a+1$ and let us now see the implications of this equality when considering transformation γ_2 (Figure 3). Then $2b-a=2(2a+1)-a=3a+2$. If $3a+2=12$ then $a=-1$ which is contrary to our assumptions. Also, $2b-2a-1=2a+1$ and if $2a+1=12$ then $a=12$ which is contrary to our assumptions. We conclude that, for $c=12$, transformation γ_2 solves the issues that are not solved by transformation γ_1.

(b) $c=11$

If $2a-b=-1$ then $b=2a+1$.

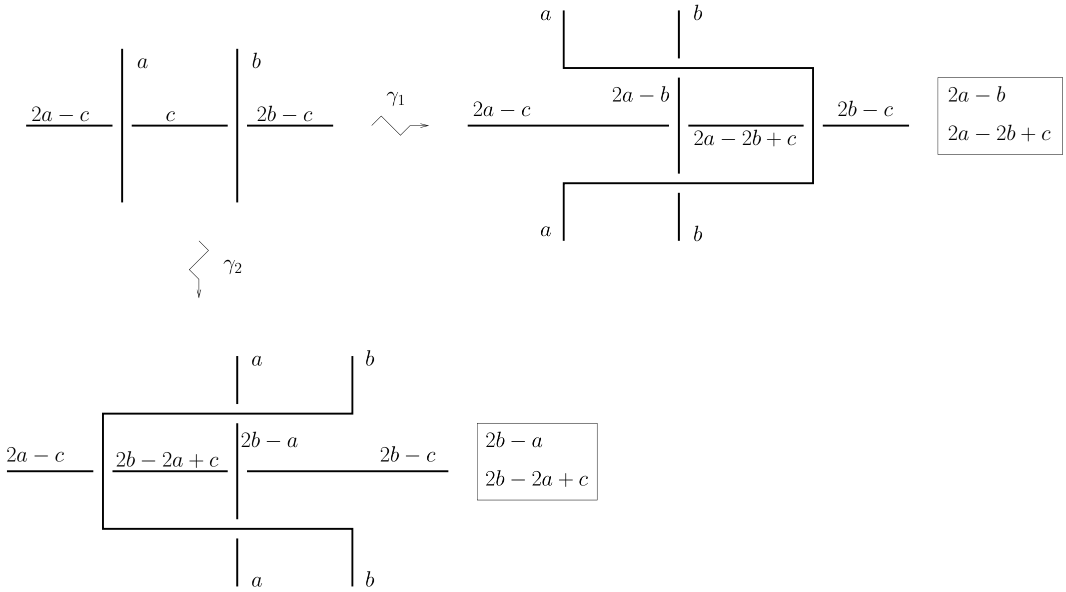

Figure 3: Transformations γ_1 and γ_2.

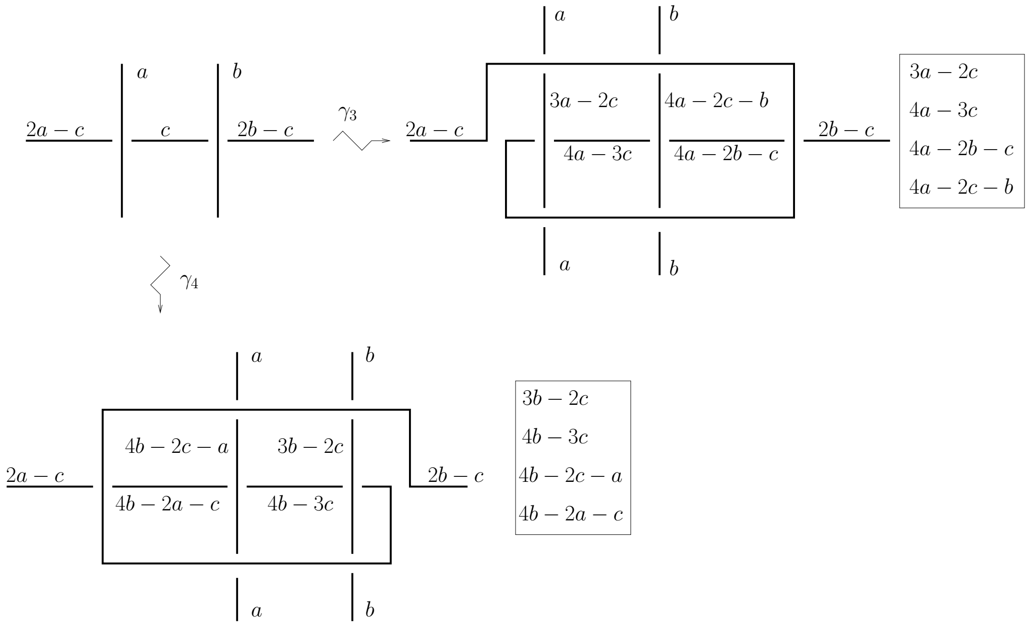

Figure 4: Transformations γ_3 and γ_4.

i. Then $2b - a = 3a + 2$ and if $3a + 2 = -1$ then $a = -1$ which contradicts the assumptions; if $3a + 2 = -2$ then $a = 3$ and $b = 2a + 1 = 7$. This $(a, b) = (3, 7)$ case is dealt with with transformation γ_3 (Figure 4).

ii. On the other hand, $2b - 2a - 2 = 2a$. If $2a = -2$ then $a = -1$ contrary to the assumptions. If $2a = 12$ then $a = 6$ and so $b = 2a + 1 = 0$. The case $(a, b) = (6, 0)$ is dealt with with transformation γ_3 (Figure 4).

If $2a - b = -2$ then $b = 2a + 2$

i. Then $2b - a = 3a + 4$. So if $3a + 4 = -2$ then $a = -2$, contrary to the assumptions. If $3a + 4 = -1$ then $a = 7$ and $b = 2a + 2 = 3$. This $(a, b) = (7, 3)$ case is dealt with with transformation γ_4 (Figure 4).

ii. On the other hand, $2b - 2a - 2 = 2a + 2$ and $2a + 2$ cannot take on 12 or 11 since it is part of the left-hand side of the transformations γ_i.

Now for $2a - 2b + c = 2a - 2b - 2$. If $2a - 2b - 2 = -2$ then $a = b$ which is contrary to the assumptions. If $2a - 2b - 2 = -1$ then $a = b + 7$.

i. Then $2b - a = b - 7$. If $b - 7 = -2$ then $b = 5$ so that $2b + 2 = 12$ which is contrary to the assumptions. If $b - 7 = -1$ then $b = 6$ so that $a = b + 7 = 0$. The case $(a, b) = (0, 6)$ is dealt with with transformation γ_4 (Figure 4).

ii. On the other hand, $2b - 2a - 2 = -14 - 2 = 10$ which does not raise any objections.

(c) The δ instance.

Consider transformation δ_1 in Figure 5.

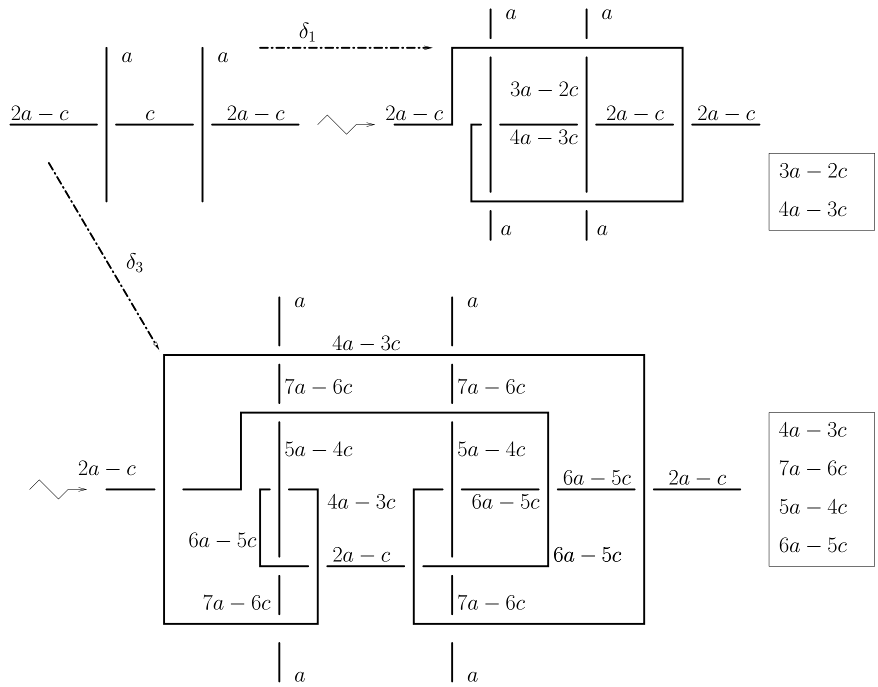

Figure 5: Transformations δ_1 and δ_3.

i. $c = 12$.
Then $3a - 2c = 3a + 2$ and if $3a + 2 = -1$ then $a = -1$ which contradicts the assumption. Also $4a - 3c = 4a + 3$ and if $4a + 3 = -1$ then $a = -1$ which contradicts the assumption.

ii. $c = 11$.
Then $3a - 2c = 3a + 4$. If $3a + 4 = -2$ then $a = -2$ which contradicts the assumptions. If $3a + 4 = -1$ then $a = 7$ and this case is dealt with with transformation δ_3 (Figure 5). Also, $4a - 3c = 4a + 6$. If $4a + 6 = -2$ then $a = -2$ which contradicts the assumptions. If $4a + 6 = -1$ then $a = 8$ and this case is dealt with with transformation δ_2 (Figure 6).

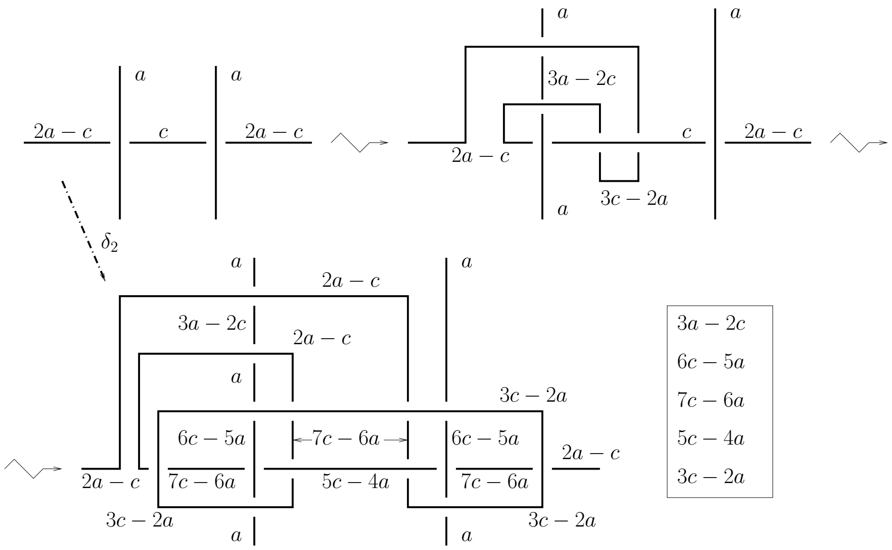

Figure 6: Transformation δ_2.

3 Part II: Elimination of colors $6, 3, 4$ and 8.

In this Section we eliminate colors $6, 3, 4$ and 8. Since we now have already removed colors 12 and 11, our technique of forcing the unwanted situations would require dealing with a significant number of cases. In addition to that, the number of transformations needed is larger than before, as we noted in the course of our work. Therefore we change the strategy and simply supply a list of extra transformations for each of the β, γ, and δ instances (since for the α instances we do not need any more transformations) and display tables from which the relevant information can be read off. These tables have 2 rows for the α, β, and δ instances. For these instances, each of these tables concern the elimination of a color c and there is only one other parameter, a, which can take on the values of the colors still available. The first row displays the different values a can take on; the second row displays the index of the transformation that settles the issue corresponding to the values of a right above this index. Here is one illustrative example. In Table 3.2, the fourth entry in the top row is 3 and the fourth entry in the bottom row is 1. This tells us that the elimination of color 6 from a polichromatic crossing whose over-arc bears color 6 and one of the under-arcs bears color 3 is realized with transformation β_1 (see Figure 2). In the same Table, the first entry in the top row is 0 and the first entry in the bottom row is X. This tells us that a polichromatic crossing whose over-arc bears color 6 and one of the under-arcs bears color 0 cannot occur since the other under-arc would then bear color $2 \cdot 6 - 0 = 12$ which has already been removed. There is a slightly different interpretation of X's in the tables for the α transformations. Since α_3 is trivially accomplished, we reserve the X's for the situations involving the α_1 or α_2 transformations where $2c - a$ has already been removed or is currently being removed.

Now for the Tables concerning the γ instances. These instances involve the elimination of a color c in a situation involving two parameters, a and b. These Tables should then be regarded as stacks of two consecutive rows. For the first pair of rows, the first row has to do with the first value parameter a can take on; the different columns correspond to the different values parameter b can take on subject to the given value of parameter a (note that $a \neq b$ for the γ transformations). The second row of this first pair of rows displays, in general, the indices of the transformations that settle the issue corresponding to the the value of b right above it along with the given value of a. Analogously for the other pairs of rows. An analogous interpretation applies to the X's as before. They correspond to $2a - c$ or $2b - c$ taking on colors which have already been eliminated or are being eliminated.

The reader will have to check that the linear combinations associated to a given transformation do not evaluate to one of the colors already eliminated or being eliminated. This seems preferable to obstructing the article with evaluating all these expressions - which the reader would still have to check.

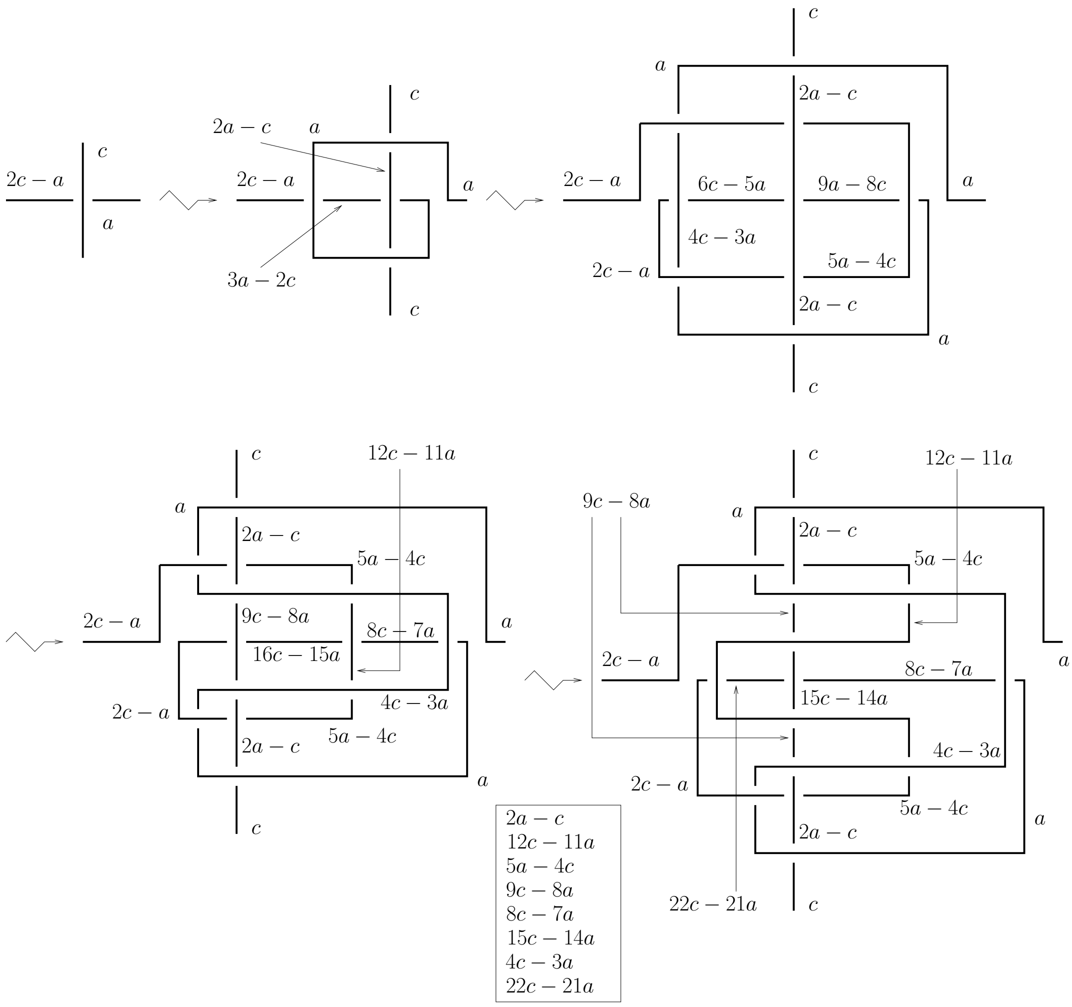

Figure 7: Transformation β_3: from the leftmost diagram in the upper row to the rightmost diagram in the lower row.

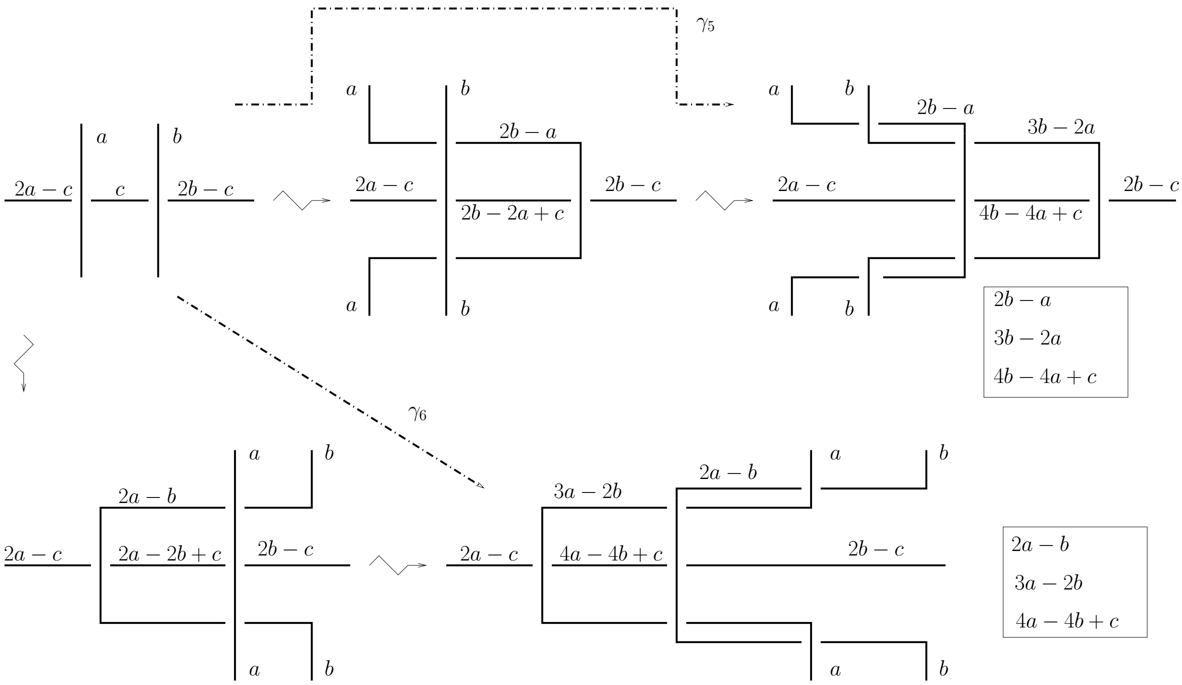

Figure 8: Transformations γ_5 and γ_6.

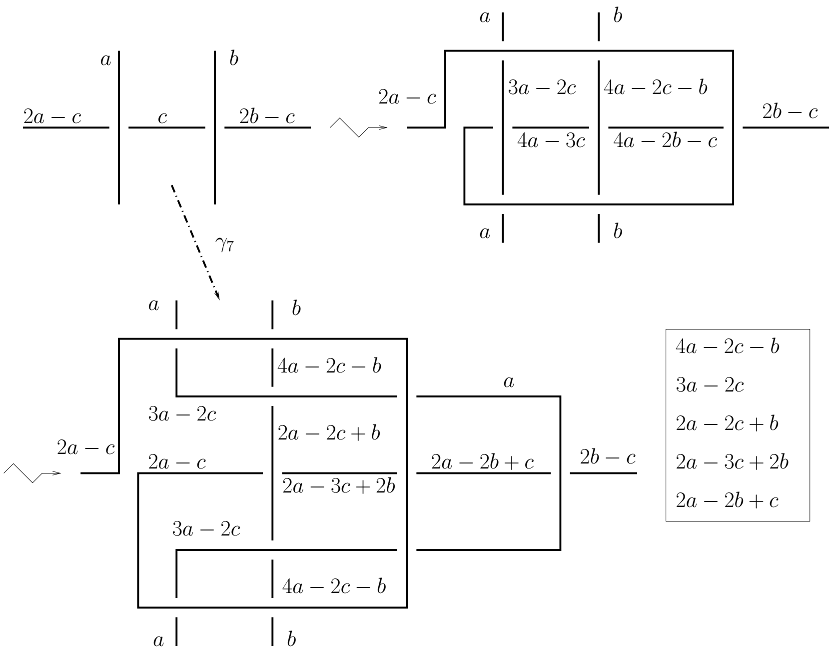

Figure 9: Transformation γ_7.

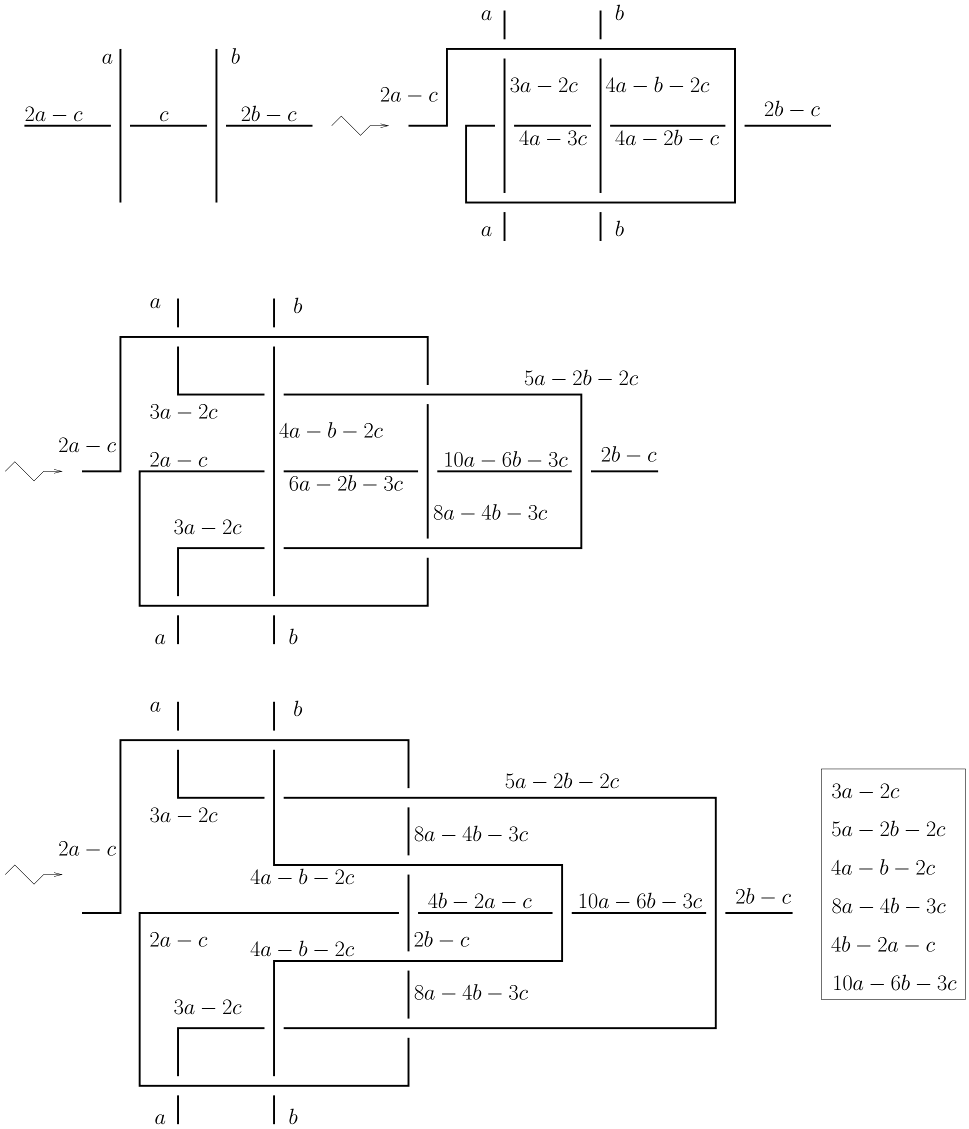

Figure 10: Transformation γ_8.

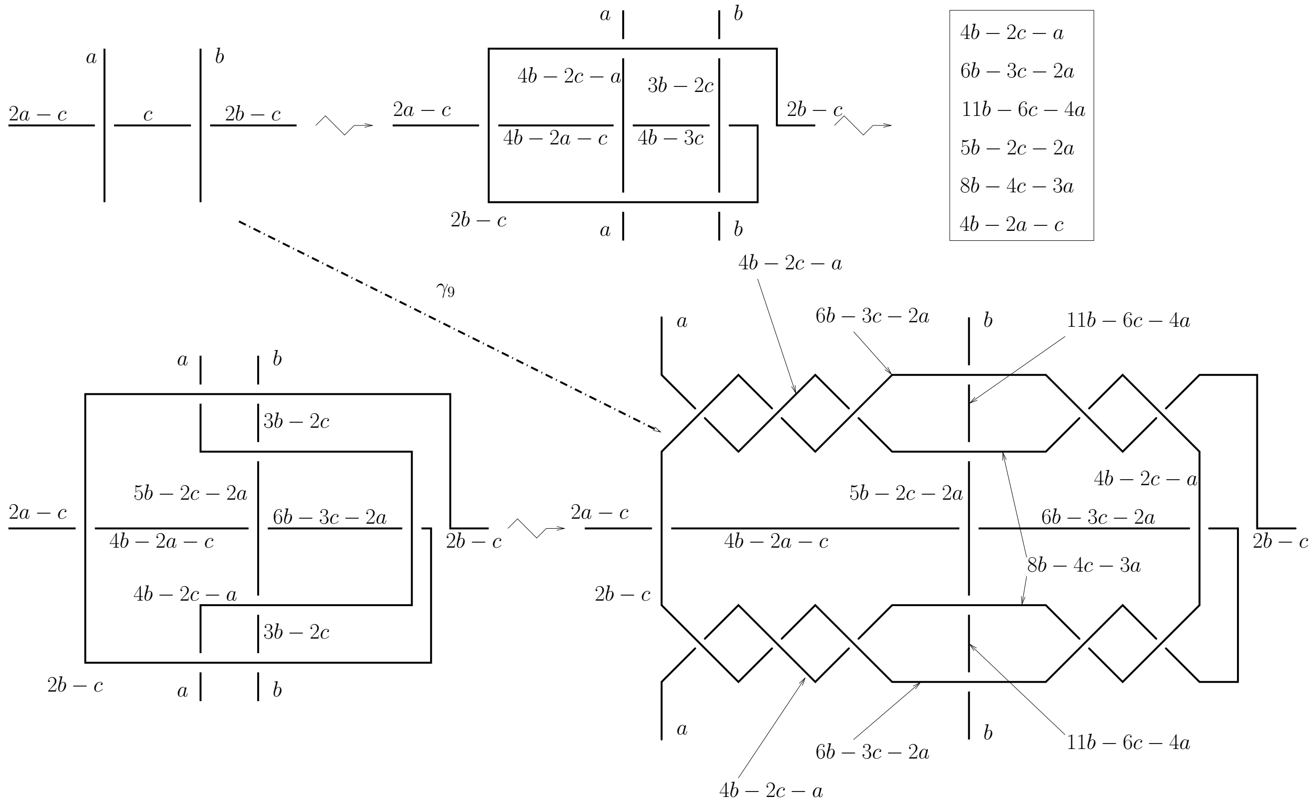

Figure 11: Transformation γ_9.

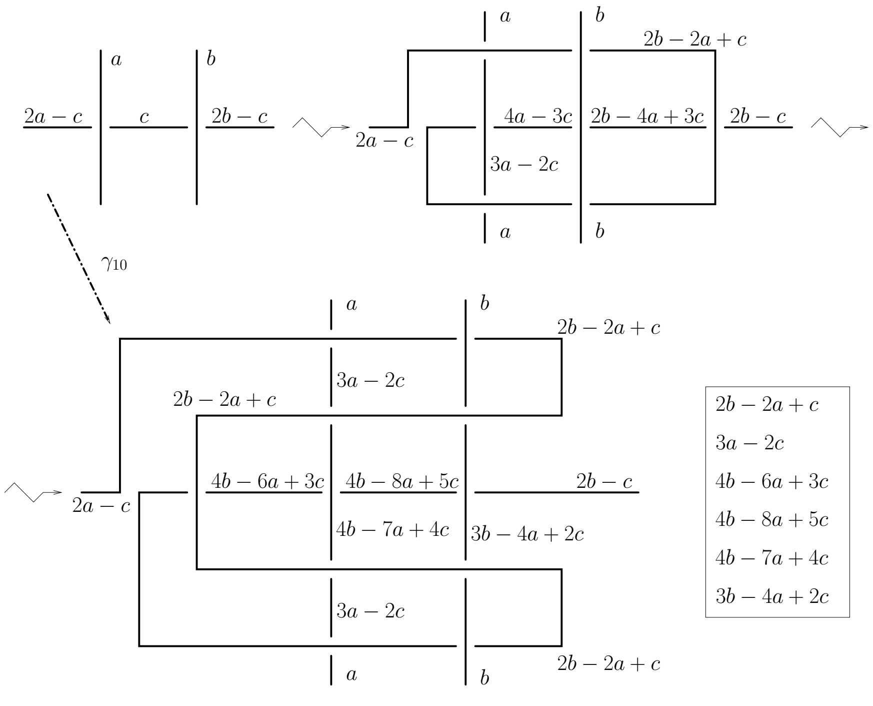

Figure 12: Transformation γ_{10}.

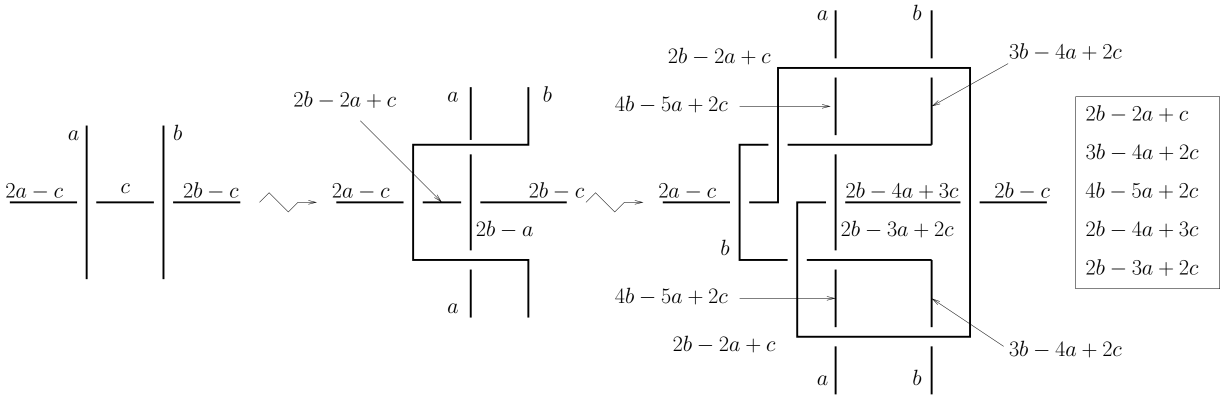

Figure 13: Transformation γ_{11}.

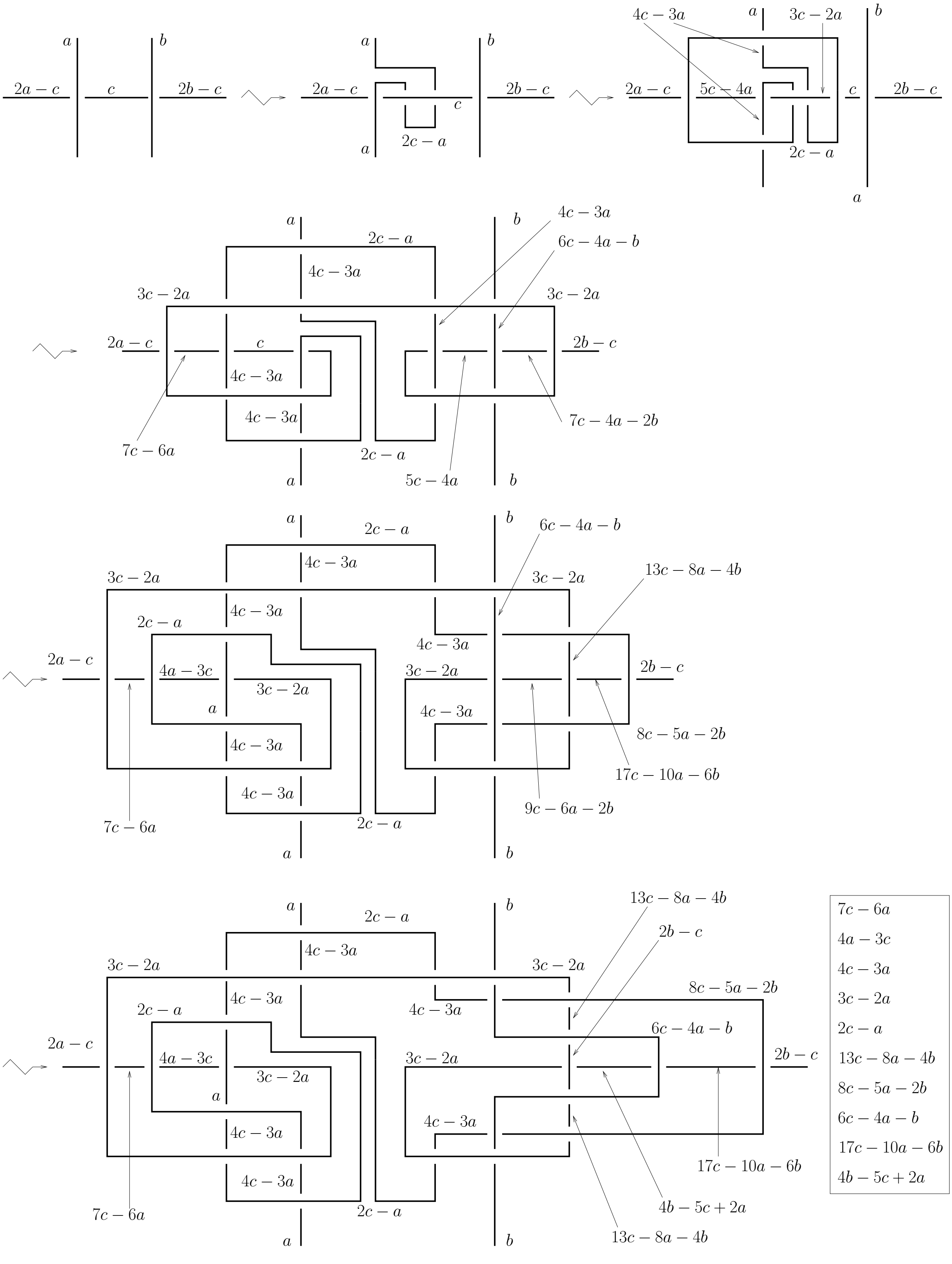

Figure 14: Transformation γ_{12}.

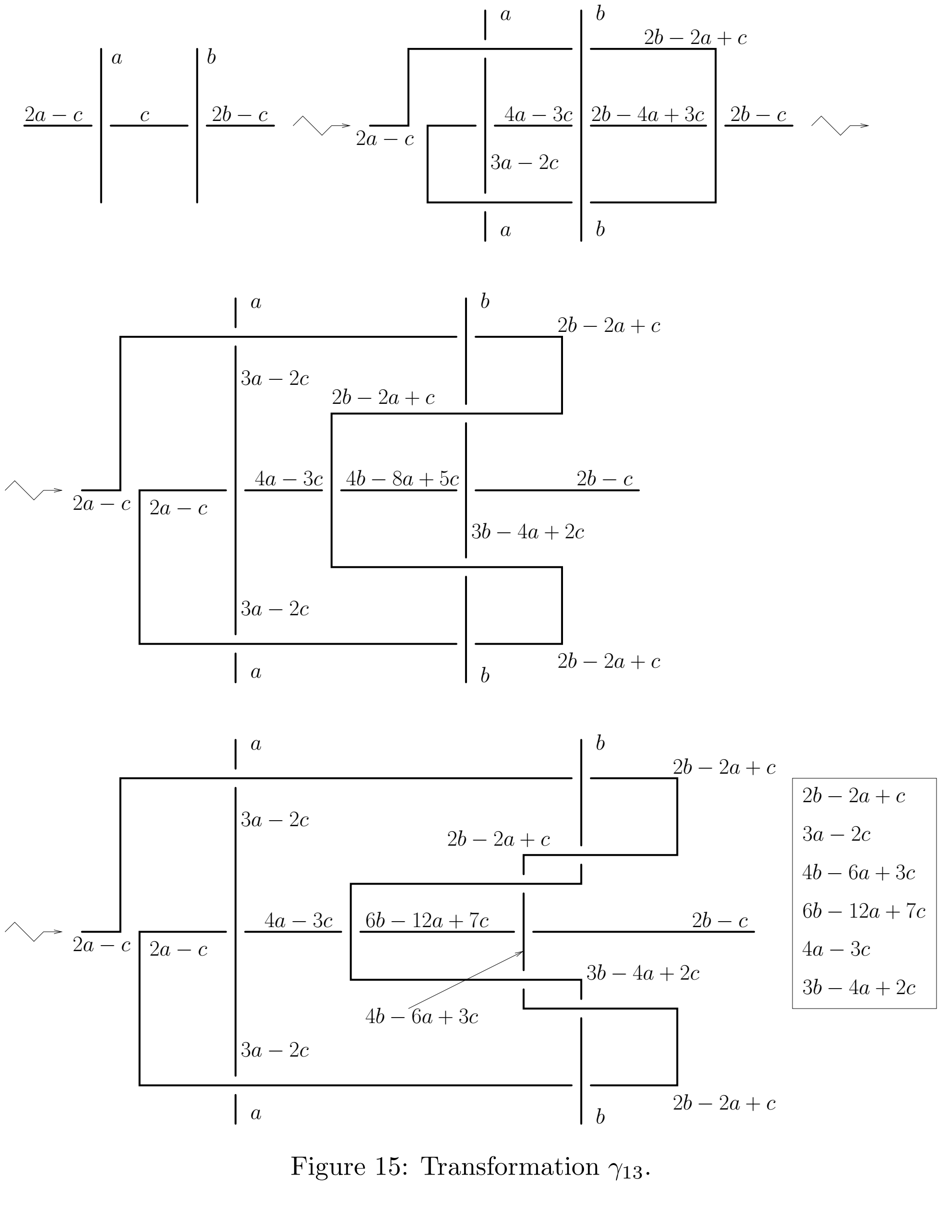

Figure 15: Transformation γ_{13}.

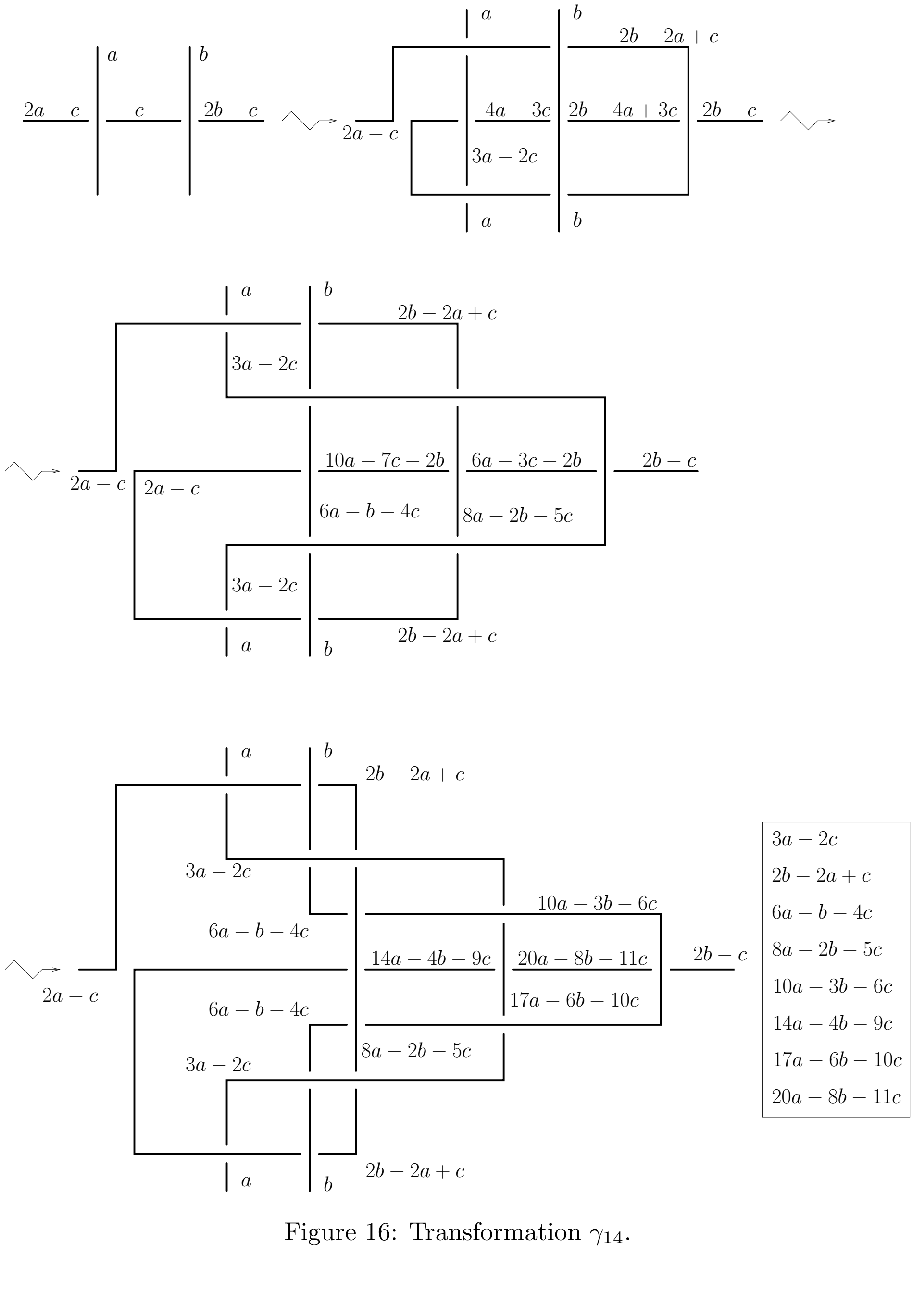

Figure 16: Transformation γ_{14}.

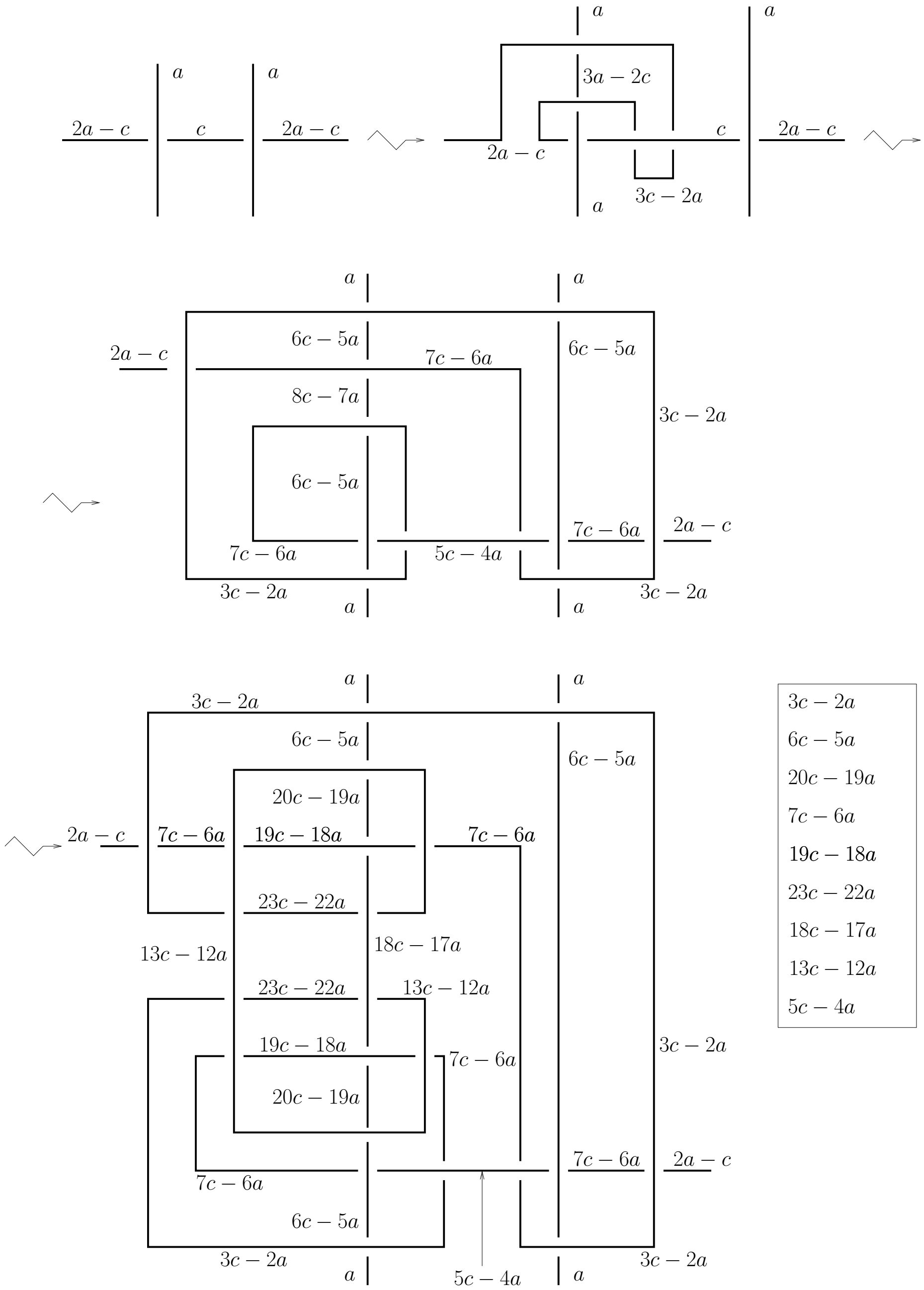

Figure 17: Transformation δ_4.

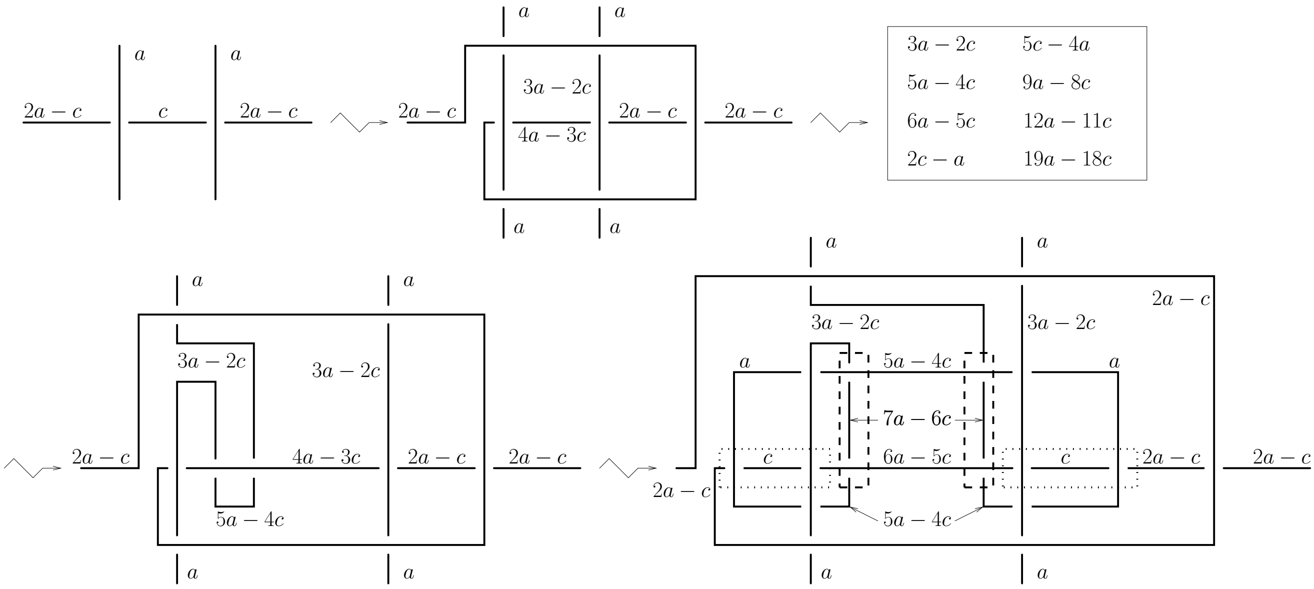

Figure 18: Transformation δ_5. The dotted boxes and broken line boxes will be addressed in Figures 19 and 20, respectively.

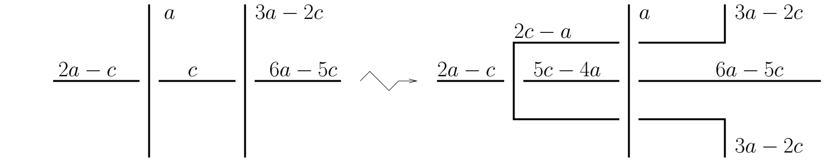

Figure 19: Transformation δ_5: addressing the dotted box from Figure 18.

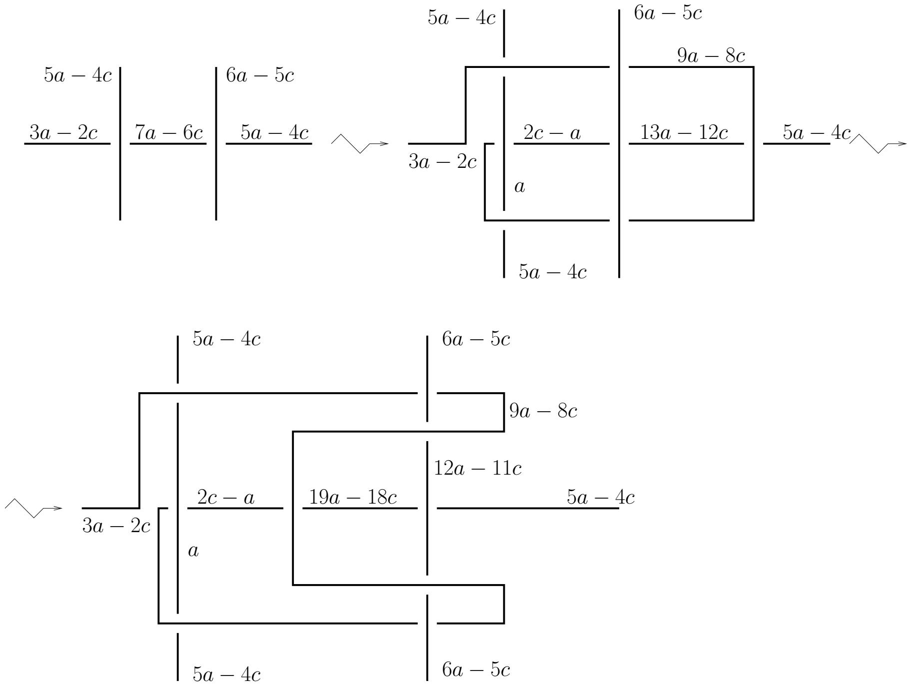

Figure 20: Transformation δ_5: addressing the broken line box from Figure 18.

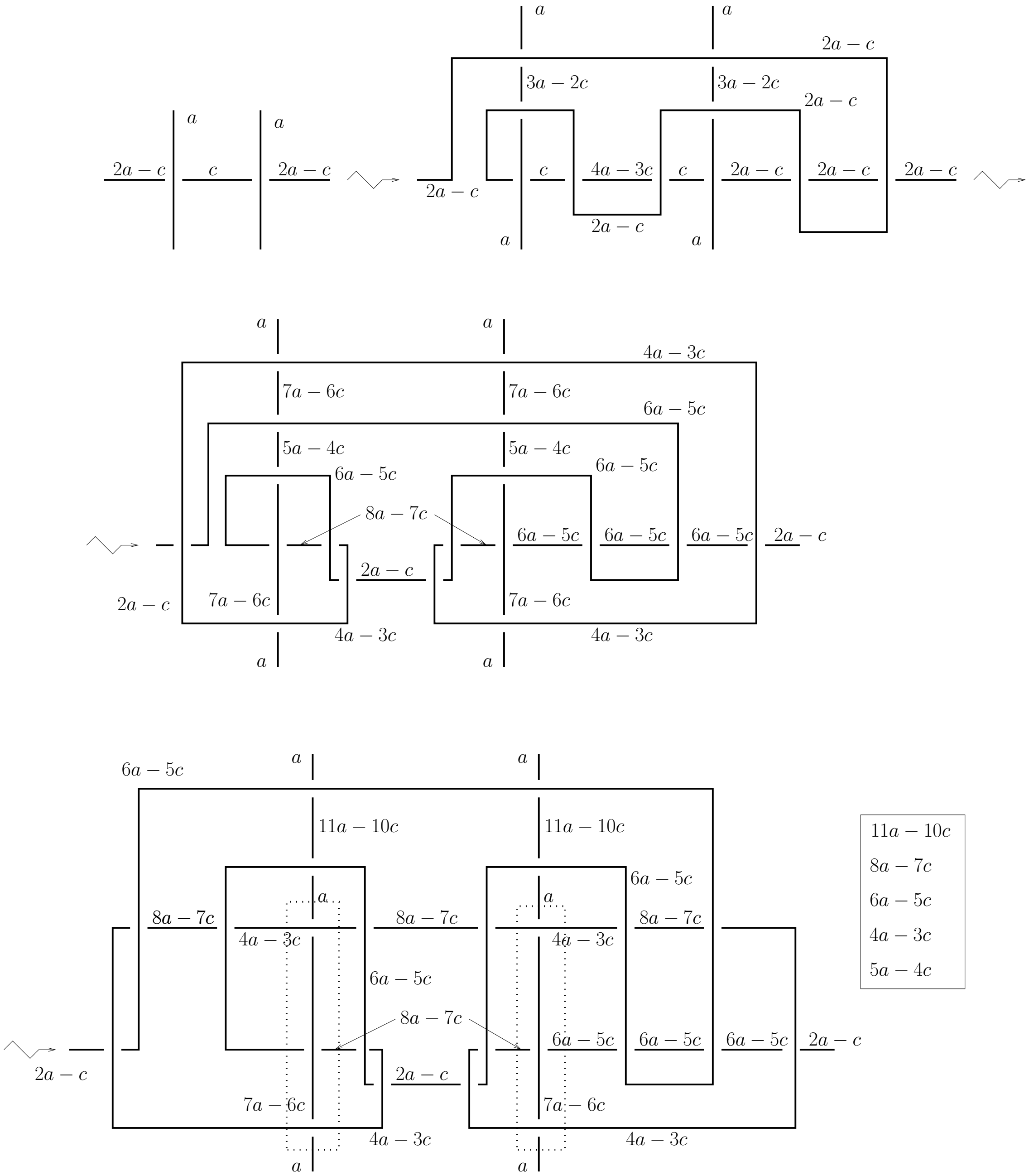

Figure 21: Transformation δ_6. The dotted boxes will be addressed in Figure 22.

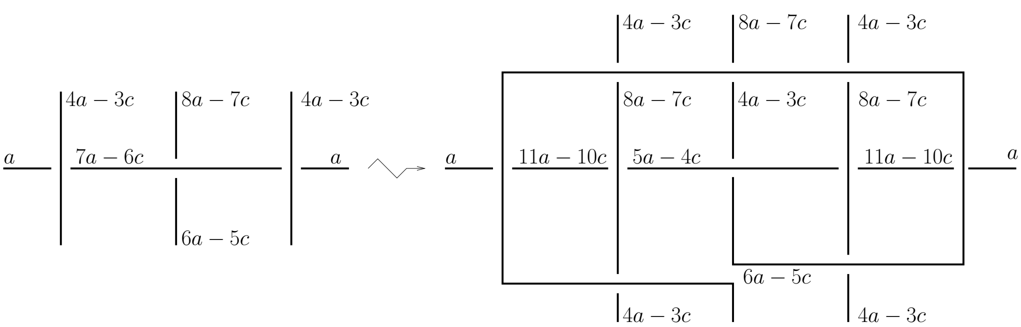

Figure 22: Transformation δ_6: addressing the dotted boxes from Figure 21.

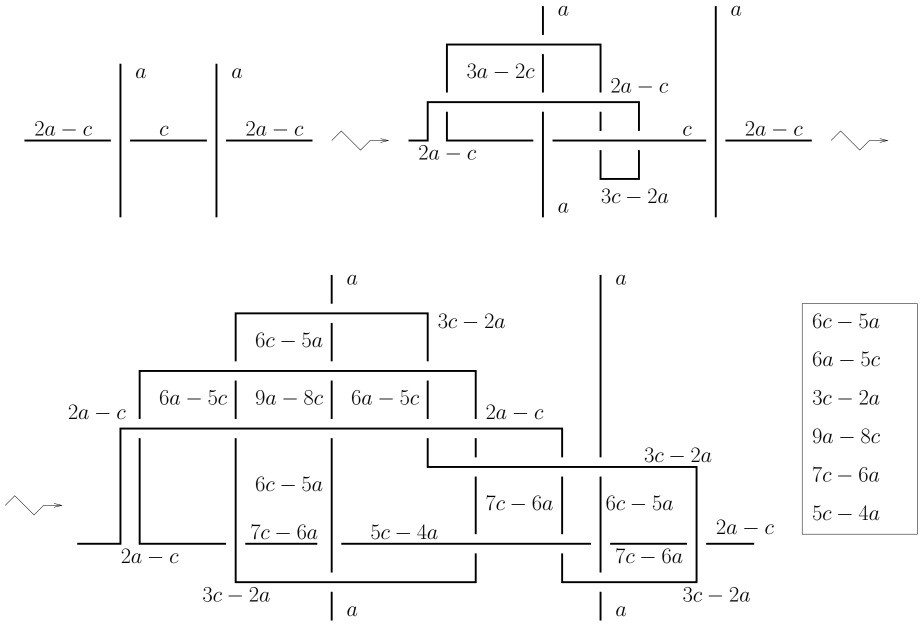

Figure 23: Transformation δ_7.

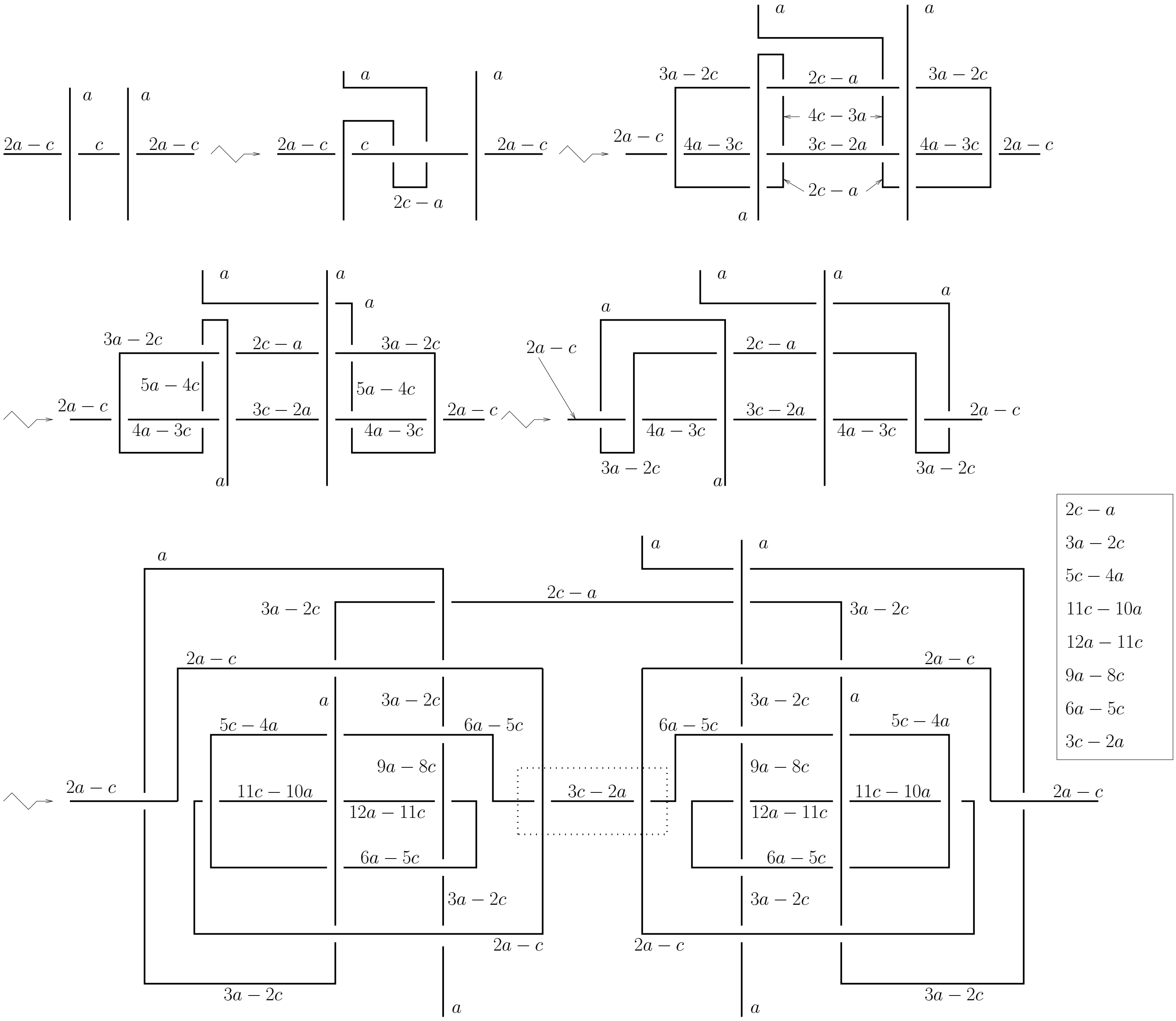

Figure 24: Transformation δ_8.

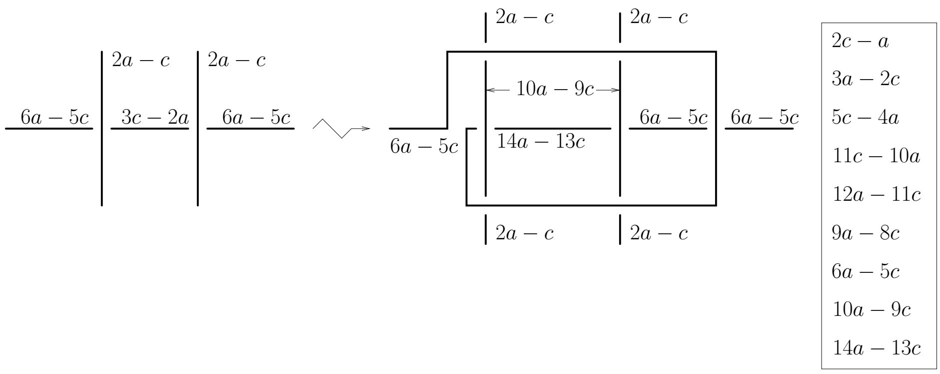

Figure 25: Transformation δ_9: it is obtained from transformation δ_8 treating its dotted box as in the current figure. The list presented in this figure is the complete list corresponding to the full transformation δ_9.

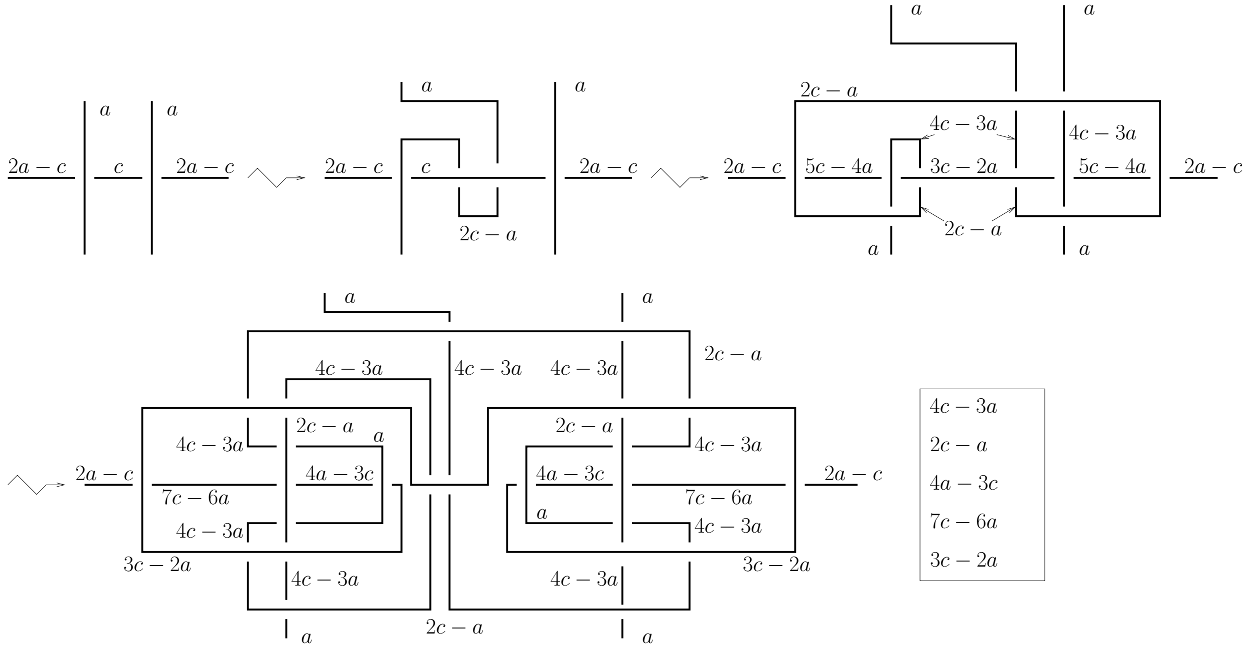

Figure 26: Transformation δ_{10}.

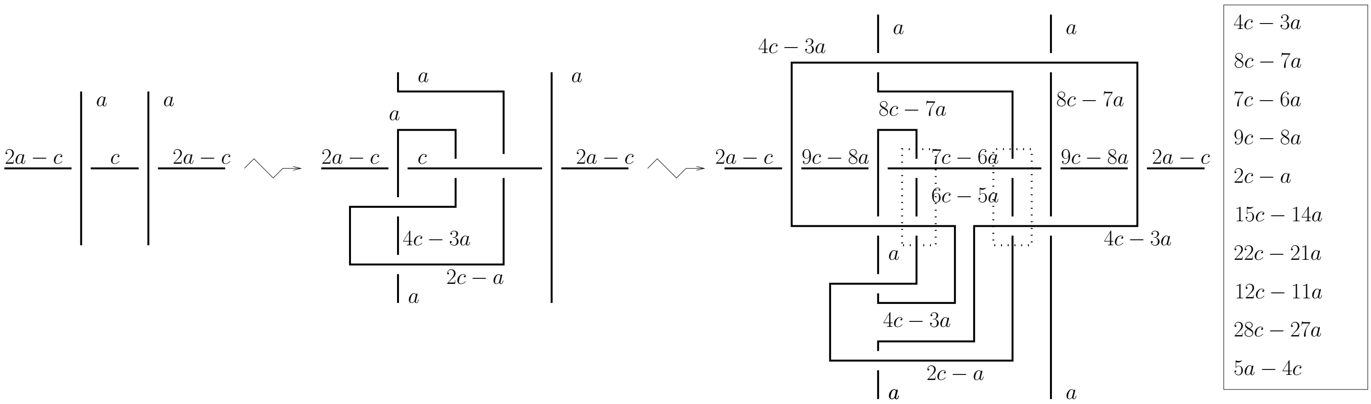

Figure 27: Transformation δ_{11}. The dotted box will be addressed in Figure 28.

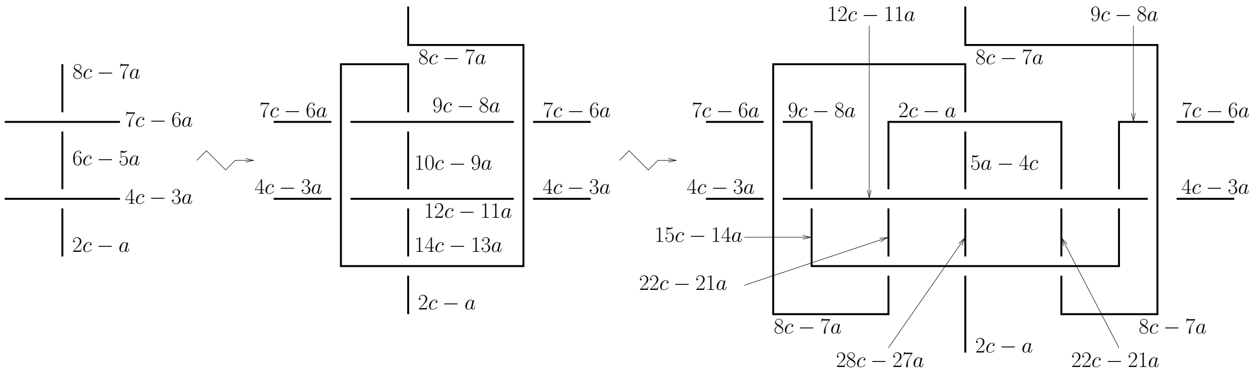

Figure 28: Transformation δ_{11}. Addressing the dotted box from Figure 27.

a	0	1	2	3	4	5	7	8	9	10
Transf. $\alpha_{\dots}$	X	X	2	1	1	1	1	1	2	1

Table 3.1: Elimination of color 6 from monochromatic crossings.

a	0	1	2	3	4	5	7	8	9	10
Transf. $\beta_{\dots}$	X	X	2	1	1	1	1	2	2	1

Table 3.2: Elimination of color 6 from over-arcs of polichromatic crossings.

$a = 0$	b	1	2	3	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	2	X	1	2	1	2	1	X	2
$a = 1$	b	0	2	3	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	1	X	2	5	2	1	1	X	1
$a = 2$	b	0	1	3	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X	X	X
$a = 3$	b	0	1	2	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	2	1	X	1	1	4	2	X	1
$a = 4$	b	0	1	2	3	5	7	8	9	10
	Transf. $\gamma_{\dots}$	1	6	X	1	1	1	6	X	2
$a = 5$	b	0	1	2	3	4	7	8	9	10
	Transf. $\gamma_{\dots}$	1	1	X	1	2	1	1	X	1
$a = 7$	b	0	1	2	3	4	5	8	9	10
	Transf. $\gamma_{\dots}$	1	1	X	3	2	1	2	X	1
$a = 8$	b	0	1	2	3	4	5	7	9	10
	Transf. $\gamma_{\dots}$	1	1	X	1	5	2	1	X	4
$a = 9$	b	0	1	2	3	4	5	7	8	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X	X	X
$a = 10$	b	0	1	2	3	4	5	7	8	9
	Transf. $\gamma_{\dots}$	1	2	X	1	1	1	2	3	X

Table 3.3: Elimination of color 6 from under-arcs joining crossings whose over-arcs bear distinct colors.

a	0	1	2	3	4	5	7	8	9	10
Transf. $\delta_{\dots}$	1	2	X	1	2	1	1	3	X	1

Table 3.4: Elimination of color 6 from under-arcs joining crossings whose over-arcs bear the same color.

a	0	1	2	4	5	7	8	9	10
Transf. $\alpha_{\dots}$	X	2	1	1	1	X	X	1	1

Table 3.5: Elimination of color 3 from monochromatic crossings.

a	0	1	2	4	5	7	8	9	10
Transf. $\beta_{\dots}$	X	2	1	2	1	X	X	1	2

Table 3.6: Elimination of color 3 from over-arcs of polichromatic crossings.

$a = 0$	b	1	2	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	X	2	1	2	X	1	2	2
$a = 1$	b	0	2	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X	X
$a = 2$	b	0	1	4	5	7	8	9	10
	Transf. $\gamma_{\dots}$	1	X	10	2	X	1	1	1
$a = 4$	b	0	1	2	5	7	8	9	10
	Transf. $\gamma_{\dots}$	2	X	!10	12	X	1	2	11
$a = 5$	b	0	1	2	4	7	8	9	10
	Transf. $\gamma_{\dots}$	1	X	1	7	X	1	1	2
$a = 7$	b	0	1	2	4	5	8	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X	X
$a = 8$	b	0	1	2	4	5	7	9	10
	Transf. $\gamma_{\dots}$	2	X	1	2	2	X	1	9
$a = 9$	b	0	1	2	4	5	7	8	10
	Transf. $\gamma_{\dots}$	1	X	2	1	2	X	1	1
$a = 10$	b	0	1	2	4	5	7	8	9
	Transf. $\gamma_{\dots}$	1	X	2	12	1	X	!9	2

Table 3.7: Elimination of color 3 from under-arcs joining crossings whose over-arcs bear distinct colors (a and b). The "!" means the values of the a and b parameters have to be interchanged.

a	0	1	2	4	5	7	8	9	10
Transf. $\delta_{\dots}$	1	X	2	3	5	X	1	1	4

Table 3.8: Elimination of color 3 from under-arcs joining crossings whose over-arcs bear the same color.

a	0	1	2	5	7	8	9	10
Transf. $\alpha_{\dots}$	1	2	X	X	1	2	X	X

Table 3.9: Elimination of color 4 from monochromatic crossings.

a	0	1	2	5	7	8	9	10
Transf. $\beta_{\dots}$	1	2	X	X	1	2	X	X

Table 3.10: Elimination of color 4 from over-arcs of polichromatic crossings.

$a = 0$	b	1	2	5	7	8	9	10
	Transf. $\gamma_{\dots}$	X	13	X	2	X	2	X
$a = 1$	b	0	2	5	7	8	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X
$a = 2$	b	0	1	5	7	8	9	10
	Transf. $\gamma_{\dots}$	!13	X	X	1	X	6	X
$a = 5$	b	0	1	2	7	8	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X
$a = 7$	b	0	1	2	5	8	9	10
	Transf. $\gamma_{\dots}$	1	X	2	X	X	1	X
$a = 8$	b	0	1	2	5	7	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X
$a = 9$	b	0	1	2	5	7	8	10
	Transf. $\gamma_{\dots}$	1	X	5	X	2	X	X
$a = 10$	b	0	1	2	5	7	8	9
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X	X

Table 3.11: Elimination of color 4 from under-arcs joining crossings whose over-arcs bear distinct colors (a and b). The "!" means the values of the a and b parameters have to be interchanged.

a	0	1	2	5	7	8	9	10
Transf. $\delta_{\dots}$	1	X	6	X	9	X	7	X

Table 3.12: Elimination of color 4 from under-arcs joining crossings whose over-arcs bear the same color.

a	0	1	2	5	7	9	10
Transf. $\alpha_{\dots}$	X	1	1	X	2	1	X

Table 3.13: Elimination of color 8 from monochromatic crossings.

a	0	1	2	5	7	9	10
Transf. $\beta_{\dots}$	X	1	2	X	!3	3	X

Table 3.14: Elimination of color 8 from over-arcs of polichromatic crossings. The elimination of color 7 via transformation β_3 assumes the change of variables $a = 9, 2c - a = 7$.

$a = 0$	b	1	2	5	7	9	10
	Transf. $\gamma_{\dots}$	2	9	2	X	2	X
$a = 1$	b	0	2	5	7	9	10
	Transf. $\gamma_{\dots}$	1	10	1	X	8	X
$a = 2$	b	0	1	5	7	9	10
	Transf. $\gamma_{\dots}$	!9	!10	11	X	12	X
$a = 5$	b	0	1	2	7	9	10
	Transf. $\gamma_{\dots}$	1	2	!11	X	1	X
$a = 7$	b	0	1	2	5	9	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X
$a = 9$	b	0	1	2	5	7	10
	Transf. $\gamma_{\dots}$	1	!8	!12	2	X	X
$a = 10$	b	0	1	2	5	7	9
	Transf. $\gamma_{\dots}$	X	X	X	X	X	X

Table 3.15: Elimination of color 8 from under-arcs joining crossings whose over-arcs bear distinct colors (a and b). The "!" means the values of the a and b parameters have to be interchanged.

a	0	1	2	5	7	9	10
Transf. $\delta_{\dots}$	1	8	10	4	X	11	X

Table 3.16: Elimination of color 8 from under-arcs joining crossings whose over-arcs bear the same color.

4 Part III: Elimination of colors 9 and 2.

In this Section we eliminate colors 9 and 2. Since we have already eliminated six colors ($12, 11, 6, 3, 4$, and 8) the most frequent symbol in the even numbered rows of the Tables in this Section is X. There are, also, four δ instances which cannot be dealt with the way we did in the preceding section. Instead we have to look into the colors at issue in order to produce adequate transformations. These instances are denoted D_1, D_2, D_3, and D_4 in Tables 4.4 and 4.9 below and presented in the figures in this Section.

Since these D_i transformations involve a different approach, a few words are in order here. D_1 and D_2 have to do with the δ instance of the elimination of color 9, whereas D_3 and D_4 have to do with the δ instance of the elimination of color 2. D_1 and D_2 (D_3 and D_4, respect.) are the very last cases to be resolved in the elimination of color 9 (color 2, respect.). Figure 29 displays transformation D_1. It has to do with the elimination of color 9 from an under-arc between two over-arcs colored both with color 5. Transformation D_1 accomplishes this by reducing the problem to the elimination of color 9 from an under-arc between two over-arcs, one colored 5 and the other colored 7. This situation has been resolved before - see Table 4.3, Transformation γ_{14} - so the elimination of color 9 from an under-arc between two over-arcs colored with 5 is accomplished.

Now for transformation D_2, which realizes the elimination of color 9 from an under-arc between two over-arcs colored 7. We start by realizing what the possibilities are for the over-arcs colored 7. These over-arcs have to eventually end up at a polichromatic crossing for otherwise there would be a split component colored with 7, and the link would have 0 determinant which contradicts our assumptions. Since the only colors available now are $0, 1, 2, 5, 7, 10$ the possibilities for the triplets $\{a, b, c\}$ from this set that realize $2b = a + c \bmod 13$ are displayed in Table 4.5. There are thus two possibilities for an over-arc colored with 7. Either the 7 ends up at a crossing whose over-arc is colored 2 and the other under-arc is colored 10; or the over-arc is colored 10 and the other under-arc is colored 0. In Figure 30 we show that other possibilities for colors at under-arcs compliant with the 7 on the over-arc before it ends up at a polichromatic crossing do not impede the progress of color 10. The role of color 10 in the elimination of color 9 is shown in Figures 31 and 32. They show the elimination of color 9 from an under-arc between two over-arcs colored with 7 is accomplished.

Now for transformation D_3, which realizes the elimination of color 2 from an under-arc between two over-arcs colored 1. This over-arc colored 1 has to eventually end up at a polichromatic crossing. There is only one possibility for this polichromatic crossing: its over-arc is colored 7 and the other under-arc is colored 0, see Table 4.10. Figure 33 shows us that other crossings that the 1 may be an over-arc to, do not impede the progression of the 7 to a convenient neighborhood of the 2 we would like to eliminate. Then Figure 34 shows how to eliminate color 2 from the situation at issue.

Finally, for transformation D_4, which realizes the elimination of color 2 from an under-arc between two over-arcs colored 10. We argue as follows. There has to be an arc colored 1 somewhere in the diagram under study. By performing Reidemeister type II moves we bring the 1 to a convenient neighborhood of the 2 we want to eliminate. We do eliminate the 2 with the help of the 1 since the colors available besides 1 are $0, 5, 7, 10$. As a matter of fact, we can see in Table 4.10 that a 1 going over 5 produces a 10 and a 1 going over a 10 produces a 5 and these colors are all admissible. A 1 going over a 0 gives rise to a 2 and we saw in Figure 34 how to deal with this situation. Finally, a 1 going over a 7 gives rise to an 8 and we deal with this situation using Transformation δ_8, see Figure 24 and Table 3.16, with $a = 1$ and $c = 8$. When we list the linear expressions corresponding to Transformation δ_8 and evaluated at $a = 1$ and $c = 8$ we obtain three colors outside the set $\{0, 1, 5, 7, 10\}$:

$$2c - a = 2 \qquad 12a - 11c = 2 \qquad 3c - 2a = 9$$

But this $2c - a = 2$ in the final diagram of Transformation δ_8 corresponds to an under-arc colored 2 between two over-arcs colored 1 - and this has been resolved in Figure 34 with Transformation D_3. As for $12a - 11c = 2$, it corresponds to an under-arc colored 2 between an over-arc colored 1 and an over-arc colored 10 and this has been dealt with (see Table 4.8). Finally, $3c - 2a = 9$ corresponds to an over-arc colored 9 between two over-arcs colored 7 and this has also been dealt with (see Figure 31). Then Figure 37 shows how to proceed in the elimination of color 2 from an under-arc between two over-arcs colored 10. This concludes the proof of Theorem 1.1.

a	0	1	2	5	7	10
Transf. $\alpha_{\dots}$	2	X	X	1	X	X

Table 4.1: Elimination of color 9 from monochromatic crossings.

a	0	1	2	5	7	10
Transf. $\beta_{\dots}$	2	X	X	1	X	X

Table 4.2: Elimination of color 9 from over-arcs of polichromatic crossings.

$a = 0$	b	1	2	5	7	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X
$a = 1$	b	0	2	5	7	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X
$a = 2$	b	0	1	5	7	10
	Transf. $\gamma_{\dots}$	X	X	X	X	X
$a = 5$	b	0	1	2	7	10
	Transf. $\gamma_{\dots}$	X	X	X	14	X
$a = 7$	b	0	1	2	5	10
	Transf. $\gamma_{\dots}$	X	X	X	!14	X
$a = 10$	b	0	1	2	5	7
	Transf. $\gamma_{\dots}$	X	X	X	X	X

Table 4.3: Elimination of color 9 from under-arcs joining crossings whose over-arcs bear distinct colors (a and b). The "!" means the values of the a and b parameters have to be interchanged.

a	0	1	2	5	7	10
Transf. $\delta_{\dots}$	X	X	X	D_1	D_2	X

Table 4.4: Elimination of color 9 from under-arcs joining crossings whose over-arcs bear the same color.

color on over-arc	0	1	2	5	7	10
colors on under-arcs		$\{0,2\}$ $\{5,10\}$	$\{7,10\}$	$\{0,10\}$	$\{0,1\}$	$\{0,7\}$ $\{2,5\}$

Table 4.5: For each color b on the top row, the duplet(s) under it displays the colors $\{a, c\}$ from $\{0, 1, 2, 5, 7, 10\}$ on the under-arcs that satisfy $2b = a + c$, mod 13, non-trivially.

a	0	1	5	7	10
Transf. $\alpha_{\dots}$	X	X	X	2	1

Table 4.6: Elimination of color 2 from monochromatic crossings.

a	0	1	5	7	10
Transf. $\beta_{\dots}$	X	X	X	2	1

Table 4.7: Elimination of color 2 from over-arcs of polichromatic crossings.

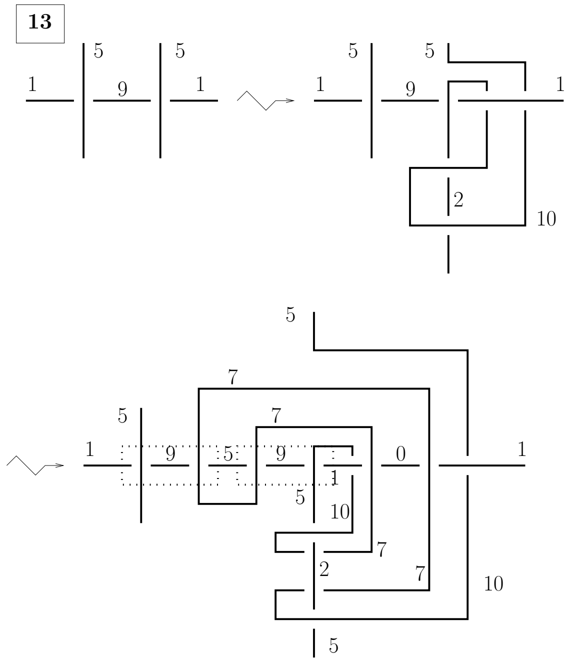

Figure 29: Transformation D_1. The issues in the dotted boxes are dealt with with transformation γ_{14}, see Table 4.3.

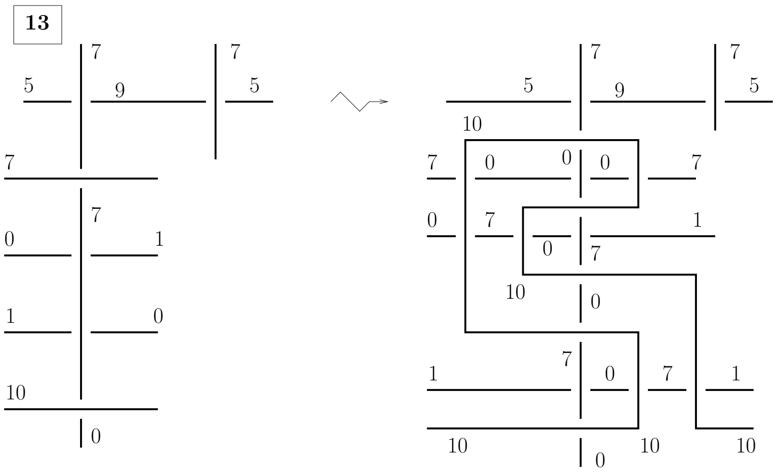

Figure 30: Preliminary considerations for Transformation D_2 below. These preliminaries allow us to disregard some complications below by showing that the 10 can progress all the way up to a convenient neighborhood of the 9.

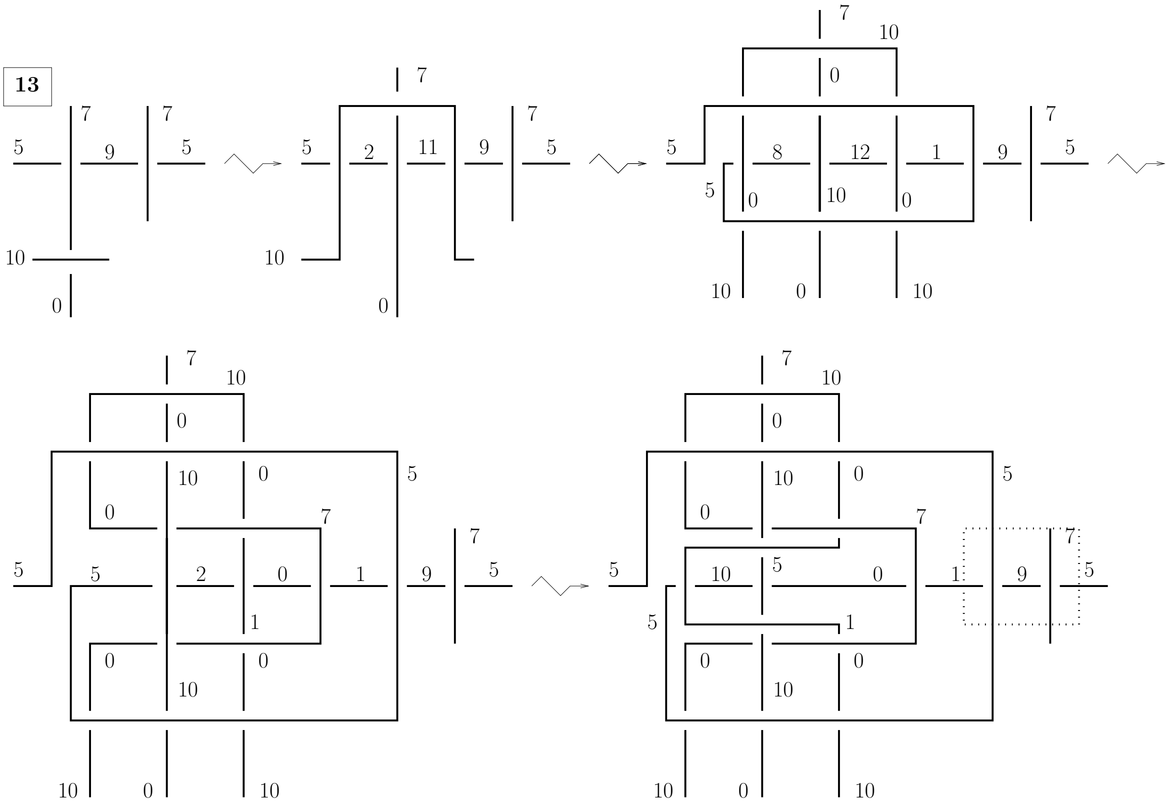

Figure 31: Transformation D_2, part I: the over-arc on the top left bearing color 7 ends up at a crossing whose over-arc bears color 10. The issue in the dotted box is dealt with with transformation γ_{14}, see Table 4.3. The last step will be useful for considerations in the sequel when removing 2 from the list of colors, see Figures 33 and 36.

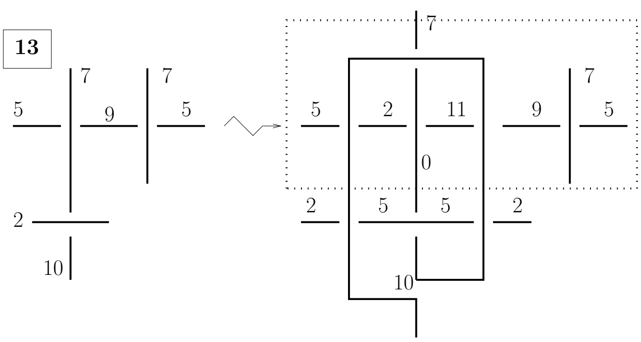

Figure 32: Transformation D_2, part II: the over-arc on the top left bearing color 7 ends up at a crossing whose over-arc bears color 2. The issue in the dotted box is dealt with with the transformation from Figure 31.

$a=0$	b	1	5	7	10
	Transf. $\gamma_{\dots}$	X	X	X	X
$a=1$	b	0	5	7	10
	Transf. $\gamma_{\dots}$	X	X	X	1
$a=5$	b	0	1	7	10
	Transf. $\gamma_{\dots}$	X	X	X	X
$a=7$	b	0	1	5	10
	Transf. $\gamma_{\dots}$	X	X	X	X
$a=10$	b	0	1	5	7
	Transf. $\gamma_{\dots}$	X	2	X	X

Table 4.8: Elimination of color 2 from under-arcs joining crossings whose over-arcs bear distinct colors (a and b).

a	0	1	5	7	10
Transf. $\delta_{\dots}$	X	D_3	X	X	D_4

Table 4.9: Elimination of color 2 from under-arcs joining crossings whose over-arcs bear the same color.

color on over-arc	0	1	5	7	10
colors on under-arc		$\{5,10\}$	$\{0,10\}$	$\{0,1\}$	$\{0,7\}$

Table 4.10: For each color b on the top row, the duplet under it displays the colors $\{a,c\}$ from $\{0,1,5,7,10\}$ on the under-arcs that satisfy $2b=a+c$, mod 13, non-trivially.

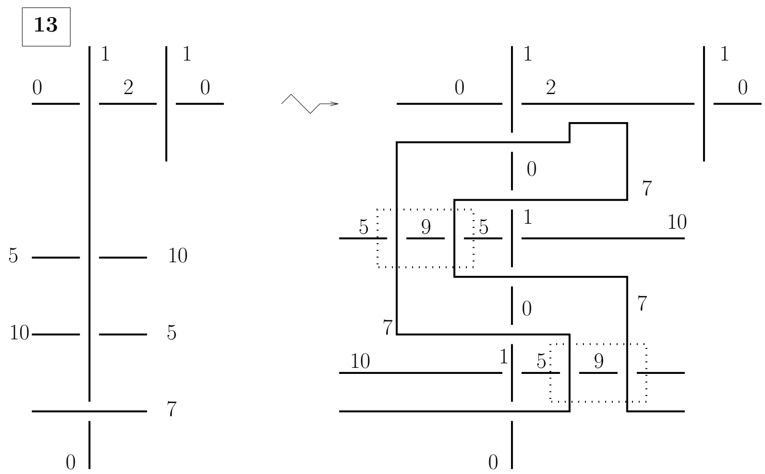

Figure 33: Preliminaries to Transformation D_3. These preliminaries allow us to disregard some complications below by showing that the 7 can progress all the way up to a convenient neighborhood of the 2. Note that the issues in the dotted boxes have been resolved before without resorting to color 2, see Figure 31. Note also that Transformation γ_{14} (Figure 16) for $a=5, b=7, c=9$ only involves $0,1,5,7,10$.

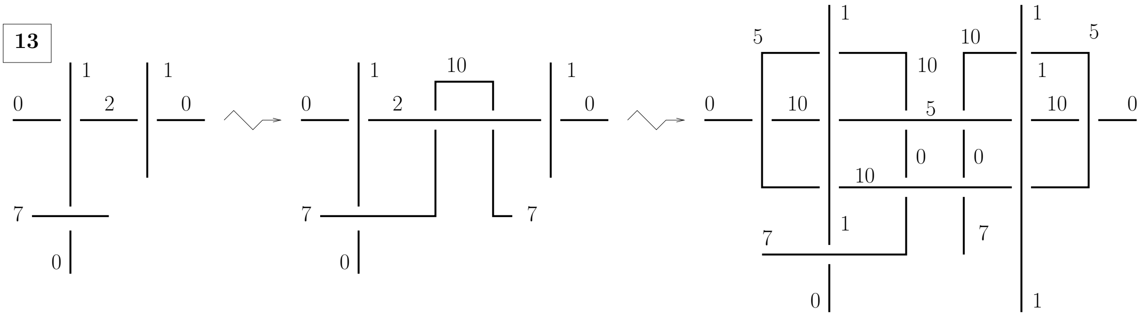

Figure 34: Transformation D_3.

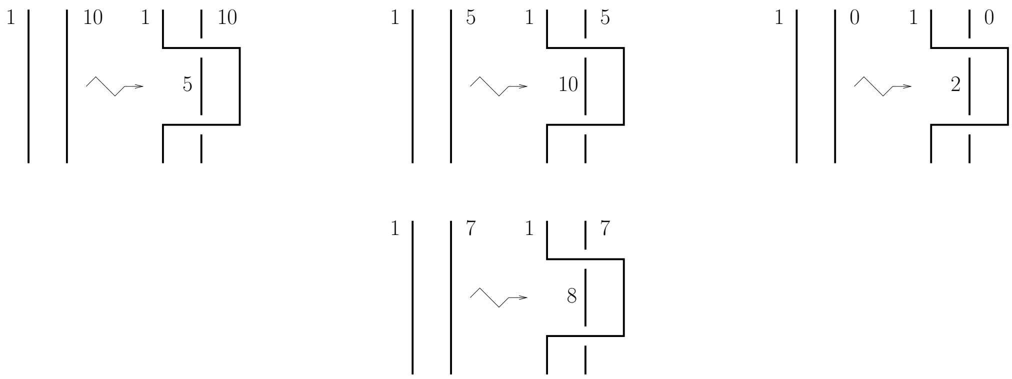

Figure 35: Preliminaries to Transformation D_4. In the first row, there are three instances of an arc colored 1 moving past arcs colored 5, 10, and 0. For the first two, the new color that shows up belongs to the set of colors available, $\{0, 1, 5, 7, 10\}$. For the last one, transformation D_3 in Figure 34 shows how to eliminate color 2 without resorting to colors outside the set $\{0, 1, 5, 7, 10\}$. In the second row the arc colored 1 moves past the arc colored 7 producing an 8. This 8 can be eliminated via Transformation δ_8 and then some more considerations, as detailed in Figure 36.

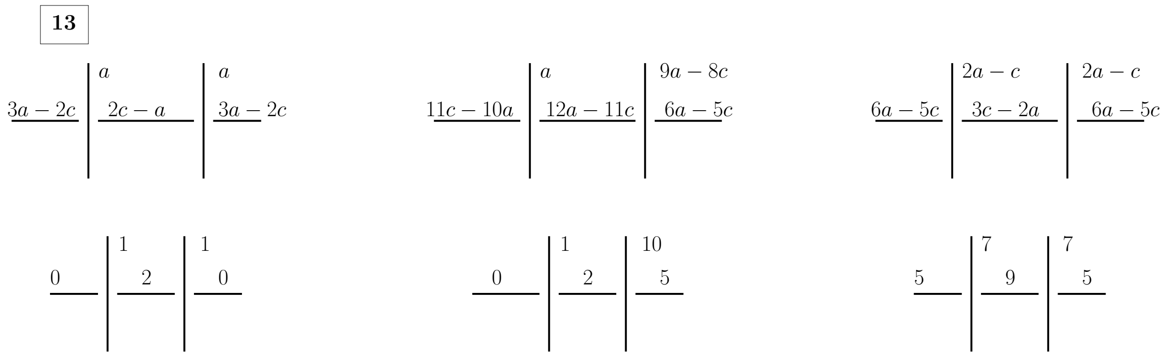

Figure 36: Preliminaries to Transformation D_4: resolving the 8 produced by having an arc colored 1 going over an arc colored 7. The three columns in this Figure show the three instances where Transformation δ_8, applied with $a = 1$ and $c = 8$, yield colors outside the set $\{0, 1, 5, 7, 10\}$, namely 2 and 9. The instance corresponding to the leftmost column is resolved as depicted in Figure 34. The instance corresponding to the middle column is resolved with Transformation γ_1, see Table 4.8. Finally, the instance corresponding to rightmost column is resolved as depicted in Figure 31. We remark that there still has to be used Transformation γ_{14}, see Figure 16, with $a = 5, b = 7, c = 9$ which eliminates color 9 resorting only to colors from $\{0, 1, 5, 7, 10\}$.

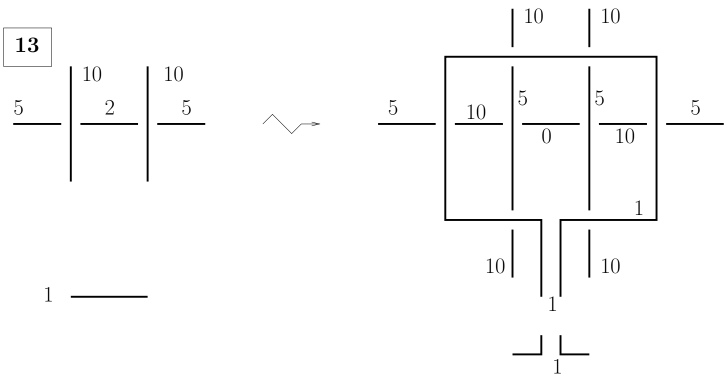

Figure 37: Transformation D_4.

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